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ABSTRACT

This is part three of a three-part manual for teachers using SMSG high school text materials. The overall purpose for each of the chapters is described and the mathematical development detailed. Background information for key concepts and answers for all exercises in each chapter are provided. Chapter topics include: (1) vectors and curves; (2) mechanics; (3) numerical analysis; (4) sequences and series; and (5) geometrical optics and waves. (MP)

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CALCULUS

PART III

SE 025 459

SCHOOL MATHEMATICS STUDY GROUP

School Mathematics Study Group

Calculus

Part 3 Teacher's Commentary

Unit 71

REVISED EDITION

The following is a list of all those who participated in the preparation of this volume:

A. Beck	Olney High School, Philadelphia, Pa.
A. A. Blank	New York University, New York, N.Y.
F. L. Elder	West Hempstead Jr., Sr. High School, N.Y.
C. E. Kerr	Dickinson College, Carlisle, Pa.
M. S. Klamkin	Ford Scientific Laboratory, Dearborn, Mich.
I. I. Kolodner	Carnegie Institute of Technology, Pittsburgh, Pa.
M. D. Kruskal	Princeton University, Princeton, N.J.
C. W. Leeds, III	Berkshire School, Sheffield, Mass.
M. A. Linton, Jr.	William Penn Charter School, Philadelphia, Pa.
H. M. Marston	Douglass College, New Brunswick, N.J.
I. Marx	Purdue University, Lafayette, Ind.
R. Pollack	New York University, New York, N.Y.
T. L. Reynolds	College of William and Mary, Williamsburg, Va.
R. L. Starkey	Cubberley High School, Palo Alto, Calif.
V. Twersky	Sylvania Electronics Defense Labs., Mt. View, Calif.
H. Weitzner	New York University, New York, N.Y.

Stanford, California
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Teacher's Commentary

Chapter 11

VECTORS AND CURVES

TC11-1. Introduction.

In this chapter we introduce the calculus of vector functions of a single real variable. For this purpose we need the elements of vector algebra and vector geometry. This preliminary material, Sections 11-1 to 11-4, can be developed without the background of calculus and it seems likely that the vector approach to geometry will find a permanent niche in the precalculus curriculum.

At this time, however, we cannot presume that this material is part of the background of any substantial number of students; therefore it is not included in the review appendices but presented in the text at the point where it is to be used.

The value of the vector approach for properties which are independent of the coordinate frame is demonstrated here by the applications to geometry and mechanics (Chapter 12) but this aspect of vector methods has extensive applications far beyond the confines of this text.

Solutions Exercises 11-1

1. Which of the following quantities are independent of the choice of coordinates?

   (a) The distance between two points.
   (b) The distance of a point from the origin.
   (c) The angle between two lines.
   (d) The angle of inclination of a line.
   (e) The area of a standard region under the graph $y = f(x)$.
   (f) The area bounded by two curves.

   The point of this question is to make the student aware of implied reference to a coordinate system. He should realize at once that the quantities described in (b), (d) and (e) are defined only in reference to a specific coordinate system while those in (a), (c), and (f) are not.

TC11-2. Vector Algebra.

Although the principal use of the concept of vector for geometrical applications is in the guise of "position vector" or directed segment with initial point at the origin, the concept of position vector is too restrictive to cover all applications. In Section 11-5, for example, we introduce the tangent vector as a function of the parameter along a curve and have no immediate reason to refer all the tangents to the same origin. Furthermore, the use of the translation model in which a vector is thought of as a mapping or operation on the set of points in space yields a natural interpretation of the addition of vectors as composition of translations, while to introduce addition of position vectors or points must seem artificial at the beginning.

In some texts the directed segment $\overrightarrow{PQ}$ is called a bound vector, bound in the sense of being "tied down" to the initial point P . The vector $\vec{V}$, by analogy, is called a free vector, free of any attachment to an initial point.

According to the dictionary (Webster-Merriam), the term "scalar" is derived from the conception of the real numbers as a linearly ordered set or "scale." In the text we prefer to stress the idea of a scalar as an operator; this is more in keeping with modern usage since the elements of any field and, in particular, the complex numbers may be scalars.

In the text we carefully maintain the distinction in notation between the point P and the vector $\vec{P}$ for the purposes of exposition. This is a nicety which may be abandoned once the distinction is clear.

Solutions Exercises 11-2

1. Let U and V be any points and $\vec{U}$, $\vec{V}$ the corresponding position vectors. In terms of $\vec{U}$ and $\vec{V}$ what vector is represented by the directed segment $\overrightarrow{UV}$?

Note. The result of this exercise is assumed later as everyday knowledge.

This corresponds to a translation $\vec{T}$ such $\vec{T}(U) = V$; hence, $\vec{T} + \vec{U} = \vec{V}$. Thus,

$$\vec{T} = \vec{V} - \vec{U} .$$

2. In Figure 11-2e, one diagonal corresponds to the sum $\vec{U} + \vec{V}$. What vector corresponds to the other diagonal?

   In the figure, take the orientation of the diagonal corresponding to $\overrightarrow{SQ}$. Then $\vec{Q} = \vec{P} + \vec{V} = [\vec{S} + (-\vec{U})] + \vec{V}$; hence, by Number 1, $\overrightarrow{SQ}$ corresponds to

   $$\vec{Q} - \vec{S} = \vec{V} + (-\vec{U}) ,$$

   That is, the difference $\vec{V} - \vec{U}$.

3. Give a geometrical justification for the inequality

   $$|\vec{U} + \vec{V}| \leq |\vec{U}| + |\vec{V}| .$$

   In Figure 11-2c, we have $\vec{W} = \vec{U} + \vec{V}$, where $\vec{U}$ and $\vec{V}$ are not collinear. The inequality merely states that the length of one side of the triangle is less than the sum of the other two. If $\vec{U}$ and $\vec{V}$ are collinear, say $\vec{U} = \lambda\vec{V}$, $|\vec{V}| > 0$, then $\vec{U} + \vec{V} = (\lambda + 1)\vec{V}$. Since $|\lambda + 1| \leq |\lambda| + |1|$, the result follows from (11).

4. Let A and B be any given points. Characterize geometrically each of the sets of points

   (a) $\{X : |\vec{X} - \vec{A}| = r\}$.

   This is the set of terminal points of the directed segments $\overrightarrow{AX}$ with initial point at A and length r . Hence, for $r > 0$ it is the sphere (or circle in the plane) of radius r ; for $r = 0$ it consists of the point A alone; for $r < 0$ it is the null set.

   (b) $\{X : |\vec{X} - \vec{A}| < r\}$.

   For $r > 0$, the interior of the sphere (or circle) in Part (a) ; otherwise, the null set.

   (c) $\{X : |\vec{X} - \vec{A}| > r\}$.

   For $r > 0$, the exterior of the sphere (or circle) in Part (a); for $r = 0$, space with the point A deleted; for $r < 0$, the whole space.

(d) $\{X : \vec{X} = \lambda\vec{A} , \lambda \text{ real}\}$.

If $A \neq O$, the point O and all points which lie in the direction of A from O or in the opposite direction, hence, the line OA . If $A = O$, the point O .

(e) $\{X : \vec{X} = \lambda\vec{A} , \lambda \geq 0\}$.

If $A \neq O$, the point O and all points which lie in the direction of A from O , hence, the ray in the direction of $\vec{A}$ with initial point at O .

(f) $\{X : \vec{X} = \vec{A} + \lambda\vec{B} , \lambda \geq 0\}$.

If $B \neq O$, the ray in the direction of $\vec{B}$ with initial point at A .

(g) $\{X : \vec{X} = \vec{A} + \lambda\vec{B} , \lambda \text{ real}\}$.

If $B \neq O$, the line through A and parallel to $\vec{B}$.

(h) $\{X : |\vec{X} - \vec{A}| = |\vec{X} - \vec{B}|\}$.

The set of points which lie at equal distances from A and B , hence the perpendicular bisector (plane or line) of the segment $\overline{AB}$.

5. For any non-null vector $\vec{A}$ obtain the unit vector (vector of length 1) in the direction of $\vec{A}$.

Let $\vec{U}$ be the desired unit vector. We have $\vec{U} = \lambda\vec{A}$ where $\lambda > 0$. From (11), under the condition of the problem,

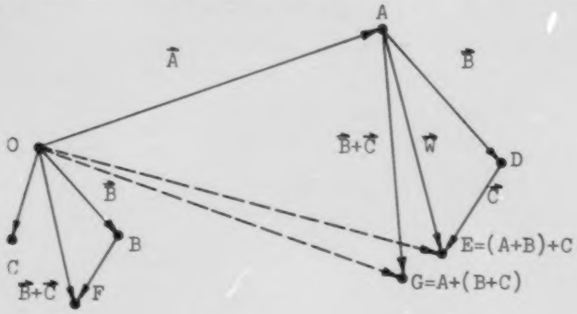
$$|\vec{U}| = |\lambda\vec{A}| = \lambda|\vec{A}| = 1 ,$$

hence, $\lambda = \frac{1}{|A|}$

$$\vec{U} = \frac{\vec{A}}{|\vec{A}|} .$$

6. Give a geometrical derivation for the associative law (6) for the addition of vectors.

In the figure, the operations indicated by the sums $(\vec{A} + \vec{B}) + \vec{C}$ and $\vec{A} + (\vec{B} + \vec{C})$ are carried out schematically. The problem is to show that the directed segment $\overrightarrow{OG}$ is the same as $\overrightarrow{OE}$. (In the figure, the points O , A , B , C are not necessarily coplanar.) It is sufficient to show that E and G are the same by proving that the vector $\vec{W}$ corresponding to $\overrightarrow{AE}$ is the same as the vector $\vec{B} + \vec{C}$ corresponding to $\overrightarrow{AG}$. But this is transparent, since $\vec{W} = \vec{C}\,\vec{B}$, that is, the composition of the translations $\vec{B}$ and $\vec{C}$ applied in that order. Thus, by definition, $\vec{W} = \vec{C} + \vec{B}$. From the parallelogram law (commutativity), $\vec{W} = \vec{B} + \vec{C}$, which completes the proof.



Alternatively, translations are functions from E^3 onto E^3. Since addition of translations amounts to composition of functions, the result is immediate from the associativity of composition.

7. From the laws of operation, A1-4, M1-2, D1-2, which define a vector space, derive the following consequences.
(a) $\lambda\vec{0} = \vec{0}$

Follow the proof of Section A1-1, Formula (8): observe that

$$\lambda\vec{0} + \lambda\vec{0} = \lambda(\vec{0} + \vec{0}) \qquad \text{(D2)},$$

$$= \lambda\vec{0} \qquad \text{(A3)};$$

that is, for $A = \lambda\vec{0}$,

$$\vec{A} + \vec{A} = \vec{A}\,; \tag{1}$$

hence,

$$(\vec{A} + \vec{A}) + (-\vec{A}) = \vec{A} + (-\vec{A}) = \vec{0} \qquad \text{(A4)};$$

whence,

$$\vec{A} + [\vec{A} + (-\vec{A})] = \vec{0} \qquad \text{(A2)} ;$$

thence,

$$\vec{A} + \vec{0} = \vec{0} \qquad \text{(A4)} ;$$

that is,

$$\vec{A} = \vec{0} \qquad \text{(A3)} ,$$

which was to be proved.

(b) $0\vec{V} = \vec{0}$

Note that

$$0\vec{V} = (0 + 0)\vec{V} = 0\vec{V} + 0\vec{V} \qquad \text{(D1)} ,$$

i.e., for $\vec{A} = 0\vec{V}$,

$$\vec{A} = \vec{A} + \vec{A} ,$$

which is the same as Equation (1) in Part (a). Then proceed as in Part (a).

(c) If $\lambda \neq 0$ and $\vec{V} \neq \vec{0}$ then $\lambda\vec{V} \neq \vec{0}$.

Suppose on the contrary, that $\lambda \neq 0$, $\vec{V} \neq 0$ and $\lambda\vec{V} = 0$. Then

$$\frac{1}{\lambda}(\lambda\vec{V}) = \frac{1}{\lambda}\vec{0} = \vec{0} \qquad \text{(by Part (a))},$$

but

$$\begin{aligned} \frac{1}{\lambda}(\lambda\vec{V}) &= (\frac{1}{\lambda} \cdot \lambda)\vec{V} && \text{(M2)} \\ &= 1\vec{V} \\ &= \vec{V} && \text{(M1)} . \end{aligned}$$

Consequently $\vec{V} = \vec{0}$, a contradiction.

(d) $(-1)\vec{V} = -\vec{V}$

Observe that

$$\begin{aligned} \vec{V} + (-1)\vec{V} &= 1\vec{V} + (-1)\vec{V} && \text{(M1)} \\ &= [1 + (-1)]\vec{V} && \text{(D1)} \\ &= 0\vec{V} = \vec{0} && \text{(by Part (b))}; \end{aligned}$$

hence $(-1)\vec{V}$ is inverse to $\vec{V}$.

(e) If $\lambda \neq 0$, the vector equation $\lambda\vec{X} + \vec{U} = \vec{V}$ has the unique solution $\vec{X} = \frac{1}{\lambda}(\vec{V} - \vec{U})$.

It is easy to verify direction that $\vec{X} = \frac{1}{\lambda}(\vec{V} - \vec{U})$ is a solution. On the other hand, if $\vec{X}$ is a solution, then

$$(\lambda\vec{X} + \vec{U}) + (-\vec{U}) = \vec{V} + (-\vec{U})$$

or

$$\begin{aligned} \lambda\vec{X} + [\vec{U} + (-\vec{U})] &= \vec{V} - \vec{U} && \text{(A2)} \\ &= \lambda\vec{X} + \vec{0} && \text{(A4)} \\ &= \lambda\vec{X} && \text{(A3)}\ ; \end{aligned}$$

hence,

$$\begin{aligned} \frac{1}{\lambda}(\lambda\vec{X}) &= \frac{1}{\lambda}(\vec{V} - \vec{U}) \\ &= (\frac{1}{\lambda} \cdot \lambda)\vec{X} && \text{(M2)} \\ &= 1\vec{X} = \vec{X} && \text{(M1)}\ , \end{aligned}$$

which proves that the only possible solution is the given one.

8. Show that the set of continuous functions on the interval $[0,1]$ is a linear vector space over the real numbers where addition and multiplication have their conventional interpretations.

The critical question is that of closure, namely, that the sum of continuous functions is a continuous function and that the product of a continuous function with a real number is continuous. The remaining vector space axioms derive immediately from the laws of algebra applied to the definitions of function and sum of functions.

Note that the set of piecewise monotone functions does not satisfy the closure properties, and that is what makes it unsuitable for a theory of the integral.

9. (a) Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be linear vector spaces over the real numbers. Show that the set of ordered pairs $\mathcal{L}_1 \oplus \mathcal{L}_2 = \{(\vec{V}_1,\vec{V}_2) : \vec{V}_1 \in \mathcal{L}_1, \vec{V}_2 \in \mathcal{L}_2\}$ is a linear vector space over the real numbers, where, for $\vec{U}_1, \vec{V}_1 \in \mathcal{L}_1$ and $\vec{U}_2, \vec{V}_2 \in \mathcal{L}_2$ addition and multiplication by a scalar in $\mathcal{L}_1 \oplus \mathcal{L}_2$ are defined by

$$(\vec{U}_1,\vec{U}_2) + (\vec{V}_1,\vec{V}_2) = (\vec{U}_1 + \vec{V}_1 , \vec{U}_2 + \vec{V}_2)$$

and

$$\lambda(\vec{V}_1,\vec{V}_2) = (\lambda\vec{V}_1,\lambda\vec{V}_2) \ .$$

The space $\mathcal{L}_1 \oplus \mathcal{L}_2$ is known as the direct sum of $\mathcal{L}_1$ and $\mathcal{L}_2$.

Closure, commutativity and associativity of addition are obvious. The null vector in $\mathcal{L}_1 \oplus \mathcal{L}_2$ is simply $(\vec{0}_1,\vec{0}_2)$ where $\vec{0}_1$ and $\vec{0}_2$ are the null vectors for $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. The inverse to $(\vec{V}_1,\vec{V}_2)$ is $(-\vec{V}_1,-\vec{V}_2)$ where $-\vec{V}_i$ is inverse to $\vec{V}_i$ in $\mathcal{L}_i$, $(i=1,2)$. The remaining laws follow similarly by direct application of the definitions of the operations.

Note that in the definition of addition of ordered pairs the sign "+" is used for three kinds of addition, the addition in $\mathcal{L}_1 \oplus \mathcal{L}_2$ which is being defined and within the parentheses, an addition in $\mathcal{L}_1$ and $\mathcal{L}_2$. Similarly, three distinct operations of scalar multiplication are involved in the definition of product.

(b) Show that $\mathcal{R}$, the real number field is a linear vector space over the real numbers, where addition and multiplication is now ordinary addition and multiplication of numbers. The set $\mathcal{R}$ considered as a linear vector space with a length defined as $|x|$ for $x \in \mathcal{R}$ is denoted by E^1 (one-dimensional euclidean space).

The linear vector space postulates here are direct consequences of the field postulates for $\mathcal{R}$ (Section A1-1).

(c) Show that euclidean two-dimensional space E^2 is given by

$$E^2 = E^1 \oplus E^1 ,$$

where length for $\vec{A} \in E^2$, given that $\vec{A} = (a,b)$ and $a , b \in E^1$, is defined by

$$|\vec{A}| = \sqrt{a^2 + b^2} \ .$$

Similarly, show that euclidean three-dimensional E^3 is given by

$$E^3 = E^2 \oplus E^1$$

where length for $\vec{V} \in E^3$, given in the form $\vec{V} = (\vec{A},c)$ with $\vec{A} \in E^2$, $c \in E^1$, is defined by

$$|\vec{V}| = \sqrt{|\vec{A}|^2 + c^2} \ .$$

This is the usual cartesian representation of E^2 with the pythagorean distance formula.

In the representation of E^3 we may write the vector $\vec{V}$ in the form

$$\vec{V} = (\vec{A},c) = ((a,b),c) = (a,b,c) ,$$

with $a , b , c \in \mathcal{R}$, and the internal parentheses are dropped as redundant information. For the length of $\vec{V}$ we have

$$|\vec{V}| = \sqrt{|\vec{A}|^2 + c^2} = \sqrt{(\sqrt{a^2 + b^2})^2 + c^2}$$
$$= \sqrt{a^2 + b^2 + c^2} ,$$

which is the pythagorean distance formula in E^3 .

TC11-3. Vector Geometry.

Solutions Exercises 11-3

1. Verify that Formula (2) gives the familiar formulas for the midpoint of a segment in terms of coordinates for P and Q .

Set $P = (x_1,y_1)$, $Q = (x_2,y_2)$ and $X = (\bar{x},\bar{y})$. Use

$$\vec{X} = \frac{1}{2}(\vec{P} + \vec{Q}) = \left(\frac{x_1 + x_2}{2} , \frac{y_1 + y_2}{2}\right)$$

to obtain from (2),

$$\bar{x} = \frac{x_1 + x_2}{2} , \bar{y} = \frac{y_1 + y_2}{2} .$$

2. Find the equation of the line through the point $P = (1,2,1)$ parallel to the position vector $(0,3,4)$; then give the coordinates of a point on the line at distance 1 from P .

The equation of the line is

$$\vec{X} = (1,2,1) + t(0,3,4) .$$

From

$$|\overline{PX}| = |\vec{X} - \vec{P}| = 5|t|$$

the two points on the line at unit distance from P are given by $t = \pm \frac{1}{5}$, hence, by $(1 , 2 \pm \frac{3}{5} , 1 \pm \frac{4}{5})$.

3. (a) Is $P = (2,1,2)$ in the plane $\mathcal{P}$ containing $A = (1,1,3)$, $B = (1,1,2)$, $C = (1,3,3)$?

The plane $\mathcal{P}$ is given by the equation

$$\vec{X} = \vec{A} + \mu(\vec{B} - \vec{A}) + \nu(\vec{C} - \vec{A})$$

or

$$\vec{X} = (1,1,3) + \mu(0,0,-1) + \nu(0,2,0) .$$

Thus P is in $\mathcal{P}$ if and only if

$$(2,1,2) = (1, 1 + 2\nu, 3 - \mu)$$

for some μ and ν, but this is clearly impossible since the x-components are distinct.

(b) Find a vector $\vec{N}$ normal to the plane $\mathcal{P}$.

A vector $\vec{N}$ is normal to $\mathcal{P}$ if and only if it is perpendicular to the vectors $\vec{B} - \vec{A}$ and $\vec{C} - \vec{A}$. Hence,

$$\vec{N} \cdot (\vec{B} - \vec{A}) = \vec{N} \cdot (\vec{C} - \vec{A}) = 0 .$$

Consequently, if $\vec{N} = (N_x, N_y, N_z)$, then

$$-N_z = 0 , \; 2N_y = 0 ,$$

and $$\vec{N} = (N_x,0,0) = N_x(1,0,0) ,$$

where N_x may be any number.

(c) Find the distance from $P = (2,1,2)$ to the plane $\mathcal{P}$.

The distance is $|\vec{P} - \vec{Q}|$ where Q is the foot of the perpendicular from P to $\mathcal{P}$. Thus Q is the point of intersection with $\mathcal{P}$ of the normal line

$$\vec{X} = \vec{P} + \lambda\vec{N} = (2,1,2) + \lambda(1,0,0)$$

where $|\lambda| = |\vec{P} - X|$. Thus Q satisfies the simultaneous conditions

$$Q = (2+\lambda,1,2) = (1,1+2\nu,3-\mu) ,$$

whence $Q = (1,1,2)$ and

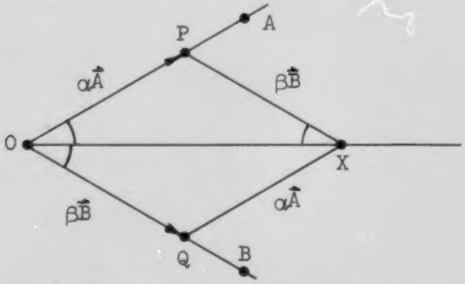
$$|\vec{P} - \vec{Q}| = |\lambda| = 1 .$$

4. Prove the corollary in Example 11-3c.

From Part (b) of Example 11-3c, there exist scalars μ_1 , ν_1 such that $\vec{Z} = \mu_1\vec{X} + \nu_1\vec{Y}$. The question is one of uniqueness. Suppose there exist other scalars μ_2 , ν_2 with, say $\nu_2 \neq \nu_1$ and $\vec{Z} = \mu_2\vec{X} + \nu_2\vec{Y}$, then $(\mu_2 - \mu_1)\vec{X} + (\nu_2 - \nu_1)\vec{Y} = 0$: Consequently $\vec{Y} = \frac{\mu_2 - \mu_1}{\nu_2 - \nu_1}\vec{X}$; that is, $\vec{X}$ and $\vec{Y}$ are collinear, a contradiction.

5. Let $\vec{A}$ and $\vec{B}$ be noncollinear vectors. Determine the equation of the ray which bisects $\angle AOB$.

Let X be any point of the ray bisecting $\angle AOB$. Then $\vec{X}$, $\vec{A}$ and $\vec{B}$



are coplanar. Therefore there exist scalars such that

(i) $$\vec{X} = \alpha\vec{A} + \beta\vec{B} .$$

The condition that X lies in the interior of the angle implies $\alpha > 0$ and $\beta > 0$. In the figure, with $\vec{P} = \beta\vec{A}$ and $\vec{Q} = \beta\vec{B}$, note that $\angle POX = \angle XOQ$ (angle bisector) and $\angle XOQ = \angle PXO$ (PX and OQ are parallel). Therefore $\triangle OPX$ is isosceles and $|\alpha A| = |\beta B|$. For simplicity, introduce the unit vectors $\frac{\vec{A}}{|\vec{A}|}$ and $\frac{\vec{B}}{|\vec{B}|}$ in the directions of $\vec{A}$ and $\vec{B}$, respectively. Set $a = \alpha|\vec{A}|$ and $b = \beta|\vec{B}|$; thus $a = b$ and (1) becomes

(2) $$\vec{X} = a\left(\frac{\vec{A}}{|\vec{A}|} + \frac{\vec{B}}{|\vec{B}|}\right), \qquad (a > 0) .$$

Conversely, any point X defined by (2) lies on the angle bisector since the figure OPXQ is a rhombus.

6. Verify that the results of (1), of Examples 11-3a, b, and of (4) do not depend on the choice of origin.

Let $\vec{W}$ be the new origin. Each vector $\vec{V}$ must be replaced by $\vec{V} - \vec{W}$ to obtain the corresponding results in the new system.

In (1) observe that

$$\begin{aligned}\vec{X} - \vec{W} &= (1 - \lambda)\vec{P} + \lambda\vec{Q} - \vec{W} \\ &= (1 - \lambda)(\vec{P} - \vec{W}) + \lambda(\vec{Q} - \vec{W}) .\end{aligned}$$

In Example 11-3a observe that the final equation from which the result follows is expressed wholly in terms of the differences of vectors, e.g., $\vec{M}_4 - \vec{M}_1$; but a difference $\vec{U} - \vec{V}$ is not affected by a change of origin,

$$\vec{U} - \vec{V} = (\vec{U} - \vec{W}) - (\vec{V} - \vec{W}) ;$$

hence, the equation holds in the new system.

In Example 11-3b, two things are involved, the equation of the median and the equation of the centroid. For the median, note that

$$\begin{aligned}\vec{X} - \vec{W} &= (1 - \lambda)\vec{P}_1 + \frac{\lambda}{2}(\vec{P}_2 + \vec{P}_3) - \vec{W} \\ &= (1 - \lambda)(\vec{P}_1 - \vec{W}) + \frac{\lambda}{2}[(\vec{P}_2 - \vec{W}) + (\vec{P}_3 - \vec{W})] ,\end{aligned}$$

and for the centroid, that

$$\begin{aligned}\vec{A} - \vec{W} &= \frac{1}{3}(\vec{P}_1 + \vec{P}_2 + \vec{P}_3) - \vec{W} \\ &= \frac{1}{3}[(\vec{P}_1 - \vec{W}) + (\vec{P}_2 - \vec{W}) + (\vec{P}_3 - \vec{W})] .\end{aligned}$$

In (4), observe that

$$\begin{aligned}\vec{R} - \vec{W} &= \vec{A} + \mu(\vec{B} - \vec{A}) + \nu(\vec{C} - \vec{A}) - \vec{W} \\ &= (\vec{A} - \vec{W}) + \mu[(\vec{B} - \vec{W}) - (\vec{A} - \vec{W})] + \nu[(\vec{C} - \vec{W}) - (\vec{A} - \vec{W})] .\end{aligned}$$

7. (a) Show that the vectors $\vec{A} = (1,1,3)$, $\vec{B} = (1,1,2)$ and $\vec{C} = (1,3,3)$ are noncoplanar.

From Example 11-3c the three vectors are coplanar if and only if there are three scalars a , b , c not all zero such that

$$a\vec{A} + b\vec{B} + c\vec{C} = 0 .$$

Insert the coordinate representations of the vectors to obtain the three scalar equations

$$a + b + c = 0 ,$$
$$a + b + 3c = 0 ,$$
$$3a + 2b + 3c = 0 .$$

These equations have only the trivial solution $a = b = c = 0$.

(b) Express the vector $\vec{D} = (2,1,2)$ in the form of a linear combination

$$\vec{D} = a\vec{A} + b\vec{B} + c\vec{C} .$$

The coefficients a , b , c are solutions of the linear system

$$a + b + c = 2 ,$$
$$a + b + 3c = 1 ,$$
$$3a + 2b + 3c = 2 ;$$

whence $a = -\frac{3}{2}$, $b = 4$, $c = -\frac{1}{2}$.

8. Show that given any four vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, $\vec{D}$, there are constants a , b , c , d , not all zero, so that

$$a\vec{A} + b\vec{B} + c\vec{C} + d\vec{D} = \vec{0} .$$

(Hint: Use the property that if a line is not parallel to a plane it must intersect that plane at precisely one point.)

If the three vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ are coplanar then by Example 11-3c we already have such a relation for the three, $a\vec{A} + b\vec{B} + c\vec{C} = \vec{0}$, where a , b , c are not all zero. Consequently,

$$a\vec{A} + b\vec{B} + c\vec{C} + 0\vec{D} = \vec{0} .$$

Suppose now that the three vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ are not coplanar, then either $\vec{D}$ is parallel to the plane ABC or the line $\vec{X} = \lambda\vec{D}$ meets it in exactly one point.

If $\vec{D}$ is parallel to ABC then there exist a directed segment R_1R_2 in ABC which represents D . Thus, in the notation of (4),

(1) $$\vec{D} = \vec{R}_2 - \vec{R}_1 = (\mu_2 - \mu_1)(\vec{B} - \vec{A}) + (\nu_2 - \nu_1)(\vec{C} - \vec{A})$$

which immediately yields a relation of the desired form.

If $\vec{D}$ is not parallel to ABC there is a number λ for which $\lambda\vec{D}$ is the position vector of a point in the plane. Furthermore $\lambda \neq 0$ since ABC does not contain the point O ($\vec{A}$, $\vec{B}$, $\vec{C}$ are not coplanar).

Consequently, by (4)

$$\lambda\vec{D} = A + \mu(\vec{B} - \vec{A}) + \nu(\vec{C} - \vec{A}) \ , \qquad (2)$$

which immediately yields the desired relation.

9. The statement of Number 8 implies that any four vectors are linearly dependent, namely that one can be expressed as a linear combination of the other three. Show from the assumption that E^3 is not contained in a plane that there do exist three linearly independent vectors in E^3, that is, vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ for which the equation

$$a\vec{A} + b\vec{B} + c\vec{C} = 0$$

is satisfied only if all three scalars a , b , c are zero.

Since E^3 is not contained in a plane there exist at least one point A other than 0 , one point B which is not collinear with A and 0 , and one point C which is not coplanar with 0 , A , and B . Since $\vec{A}$, $\vec{B}$ and $\vec{C}$ are not coplanar the condition of Example 11-3C cannot be satisfied.

10. Prove if $\vec{A}$, $\vec{B}$, $\vec{C}$ are not coplanar then any vector $\vec{Z}$ can be represented as a linear combination

$$\vec{Z} = a\vec{A} + b\vec{B} + c\vec{C}$$

and that the representation is unique.

If $\vec{Z}$ is parallel to the plane ABC then Equation (1) in the solution of Number 8 gives the desired representation. If $\vec{Z}$ is not parallel to ABC , then, since $\lambda \neq 0$ in Equation (2) of the solution of Number 8, the desired representation is obtained on division by λ . The representation is unique, for if $a_1\vec{A} + b_1\vec{B} + c_1\vec{C} = a_2\vec{A} + b_2\vec{B} + c_2\vec{C}$ then $(a_1 - a_2)\vec{A} + (b_1 - b_2)\vec{B} + (c_1 - c_2)\vec{C} = \vec{0}$. But $\vec{A}$, $\vec{B}$, $\vec{C}$ are not coplanar and, therefore, by Example 11-3c, $a_1 = a_2$, $b_1 = b_2$, and $c_1 = c_2$.

11. Show if $\vec{B}$ and $\vec{C}$ are not collinear that $\vec{X} = \vec{A} + p\vec{B} + q\vec{C}$ is the equation of a plane passing through the point $\vec{A}$ and parallel to the vectors $\vec{B}$ and $\vec{C}$. We say that a plane is parallel to a vector $\vec{V}$ if it contains a directed segment $\overrightarrow{PQ}$ which represents $\vec{V}$.

This definition of parallelism is adopted because it generalizes to spaces of any dimension. For example, in E^2 , two lines are either parallel or intersect, but in E^3 there is a third possibility: two lines may be skew. Similarly, in E^4 , a plane and a line may be skew, that is, neither parallel nor intersecting. Thus the property given as the hint in Number 8 is a characterization of three-dimensional space.

Observe first on taking $p = q = 0$ that A is a point of the set $\mathcal{P} = \{X : \vec{X} = \vec{A} + p\vec{B} + q\vec{C} , p , q \text{ real}\}$. Now if $X , Y \in \mathcal{P}$ then $\vec{Y} - \vec{X}$ is represented by the directed segment $\overrightarrow{XY}$. Suppose $\vec{X} = \vec{A} + p_1\vec{B} + q_1\vec{C}$ and $\vec{Y} = \vec{A} + p_2\vec{B} + q_2\vec{C}$, then the segment

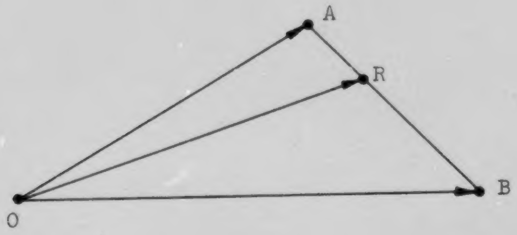
$$\overline{XY} = \{Z : \vec{Z} = (1 - \lambda)\vec{X} + \lambda\vec{Y} = \vec{A} + [1 + \lambda(p_2 - p_1)]\vec{B} + [1 + \lambda(q_2 - q_1)]\vec{C} ,$$

$$0 \leq \lambda \leq 1\} ,$$

is clearly contained in $\mathcal{P}$. Now $\vec{B}$ is represented in $\mathcal{P}$ by $\overrightarrow{XY}$ when $p_2 = 1$ and $p_1 = q_1 = q_2 = 0$, and $\vec{C}$ is represented by $\overrightarrow{XY}$ when $q_2 = 1$ and $p_1 = p_2 = q_1 = 0$ (The choice of coefficients is obviously not unique).

So far it is not determined that $\mathcal{P}$ is a plane, only that it contains A and is parallel to B and C . Observe that $\mathcal{P}$ can be expressed in the form (4) by setting $\vec{B} = \vec{B}^* - \vec{A}$, $\vec{C} = \vec{C}^* - \vec{A}$. Thus $\mathcal{P}$ is a plane if A , B^* , C^* are not collinear. But the three points are collinear if and only if $\vec{B}^* - \vec{A} = \vec{B}$ and $\vec{C}^* - \vec{A} = \vec{C}$ are collinear. Consequently, if $\vec{B}$ and $\vec{C}$ are not collinear then $\mathcal{P}$ is a plane. If $\vec{B}$ and $\vec{C}$ are collinear there are two possibilities. First, if $\vec{B} = \vec{C} = \vec{0}$ then $\mathcal{P}$ consists of the point A alone. Second, if one of the two vectors is non-null, say $C \neq 0$, then $\mathcal{P} = \{X : \vec{X} = \vec{A} + \lambda\vec{C}\}$ and $\mathcal{P}$ is a straight line (see Exercises 11-2, No. 4(g)).

12. (a) In the accompanying figure, R is any point on the line AB .



Obtain the representation $\vec{R} = a\vec{A} + b\vec{B}$, and determine $\frac{|\overline{AR}|}{|\overline{RB}|}$ in terms of a and b.

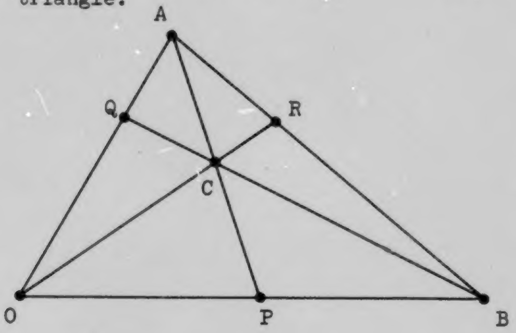
Since R is on the line AB,

$$\vec{R} = (1 - \lambda)\vec{A} + \lambda\vec{B} ;$$

then $(1 - \lambda)(\vec{R} - \vec{A}) = \lambda(\vec{R} - \vec{B})$, and $\left|\frac{\lambda}{1 - \lambda}\right|$ is the desired ratio. Since $\vec{A}$ and $\vec{B}$ are not collinear, the representation of R as a linear combination of $\vec{A}$ and $\vec{B}$ is unique. We conclude that $\lambda = b$, $1 - \lambda = a$, and

$$\frac{|\overline{AR}|}{|\overline{RB}|} = \left|\frac{b}{a}\right| .$$

Λ(b) In the accompanying figure C is any point not on a side of the triangle.



Set $\vec{R} - \vec{A} = \alpha(\vec{B} - \vec{R})$, $\vec{P} - \vec{B} = \beta(\vec{O} - \vec{P})$, and $\vec{Q} - \vec{O} = (\vec{A} - \vec{Q})$. Show that

$$\alpha\beta\gamma = 1 .$$

This result together with its converse (namely, if P, Q, R divide their respective sides so that this relation holds, then the lines AP, BQ, and OR are concurrent) is Ceva's Theorem (Giovanni Ceva, Italian 1647 - 1736).

From Part (a),

$$\vec{R} = (1 - \lambda)\vec{A} + \lambda\vec{B} .$$

and, since $\vec{C}$ is collinear with $\vec{R}$

$$\vec{C} = \sigma\vec{R} = \sigma(1 - \lambda)\vec{A} + \sigma\lambda\vec{B} .$$

Set

$$\vec{P} = \mu\vec{B} , \vec{Q} = \nu\vec{A} .$$

In terms of λ, μ, ν

$$(1) \qquad \alpha = \frac{\lambda}{1 - \lambda} \,, \; \beta = \frac{1 - \mu}{\mu} \,, \; \gamma = \frac{\nu}{1 - \nu} \,.$$

Now, for some real p ,

$$\begin{aligned} \vec{P} &= (1 - p)\vec{A} + p\vec{C} \\ &= [(1 - p) + p\sigma(1 - \lambda)]\vec{A} + p\sigma\lambda\vec{B} \,. \end{aligned}$$

Since $\vec{A}$ and $\vec{B}$ are noncollinear, the two representations of $\vec{P}$ must be the same; that is $(1 - p) + p\sigma(1 - \lambda) = 0$ and $p\sigma\lambda = \mu$. Eliminate p to obtain

$$\mu = \frac{\sigma\lambda}{1 - \sigma + \sigma\lambda} \,;$$

whence, from (1),

$$(2) \qquad \beta = \frac{1 - \sigma}{\sigma\lambda} \,.$$

Similarly, for some real q ,

$$\begin{aligned} \vec{Q} &= (1 - q)\vec{B} + q\vec{C} \\ &= q\sigma(1 - \lambda)\vec{A} + [(1 - q) + q\sigma\lambda]\,\vec{B} \,; \end{aligned}$$

hence,

$$\nu = q\sigma(1 - \lambda) \quad \text{and} \quad (1 - q) + q\sigma\lambda = 0 \,.$$

From this $\quad \nu = \dfrac{\sigma(1 - \lambda)}{1 - \sigma\lambda} \,,$ and, from (1)

$$(3) \qquad \gamma = \frac{\sigma(1 - \lambda)}{1 - \sigma} \,.$$

Multiply the expression for α in (1) by the expressions for β and γ in (2) and (3) to obtain the desired result: $\alpha\beta\gamma = 1$.

Conversely, suppose $\alpha\beta\gamma = 1$. Take C as the intersection of AP and BQ , then prove C lies on OR . From the two representations of $\vec{P}$, write $\vec{C}$ as a linear combination of $\vec{A}$ and $\vec{B}$:

$$\vec{C} = \frac{p - 1}{p}\vec{A} + \frac{\mu}{p}\vec{B} \,.$$

Similarly, from the two representations of $\vec{Q}$;

$$\vec{C} = \frac{\nu}{q}\vec{A} + \frac{q - 1}{q}\vec{B} \,.$$

Consequently,

(4) $$\vec{C} = \frac{p-1}{p}\vec{A} + \frac{q-1}{q}\vec{B}$$

and

(5) $$\mu = \frac{p(q-1)}{q}\ , \quad \nu = \frac{q(p-1)}{p}\ .$$

From (1) and (5),

$$\beta = \frac{p+q-pq}{p(q-1)}\ , \quad \gamma = \frac{q(p-1)}{p+q-pq}\ ;$$

hence, since $\alpha\beta\gamma = 1$,

$$\alpha = \frac{p}{p-1} \cdot \frac{q-1}{q}\ .$$

Consequently, from (1),

$$\vec{R} = (1-\lambda)\vec{A} + \lambda\vec{B} = (1-\lambda)(\vec{A} + \alpha\vec{B})\ .$$

But, from (4)

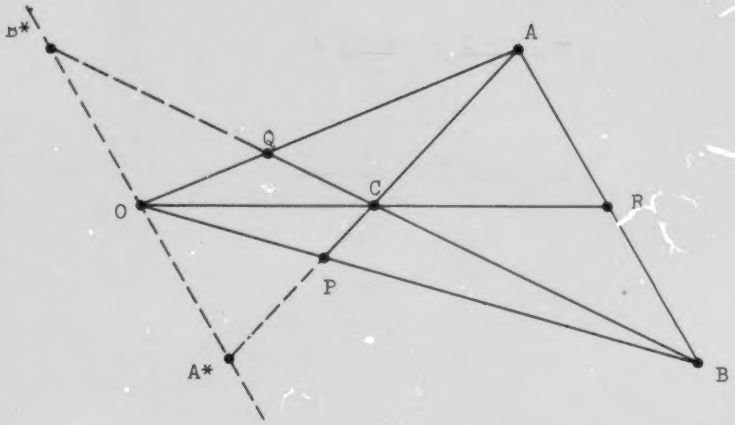
$$\vec{C} = \frac{p-1}{p}(\vec{A} + \vec{B}) = \sigma\vec{R}$$

where $\sigma = \frac{p-1}{p(1-\lambda)}$. Thus the converse is proved.

Observe in the proof of both the proposition and converse that P , Q , and R need not be interior points of their respective sides but any points of those lines other than vertices. In that case we still have $\alpha\beta\gamma = 1$ where α , β , and γ are defined by (1) but the geometrical interpretation has to be altered in replacing the lengths by signed magnitudes. Thus we take $\alpha = \frac{\overrightarrow{AR}}{\overrightarrow{RB}}$, where by this ratio we mean $\frac{|\overrightarrow{AR}|}{|\overrightarrow{RB}|}$ if the two directed segments have the same direction, and $-\frac{|\overrightarrow{AR}|}{|\overrightarrow{RB}|}$ if the directions are opposite.

Note that the result of Example 11-3b is an immediate consequence of the converse statement.

Note also the following simple synthetic proof. Through O draw a parallel to AB and extend $\overline{AP}$ and $\overline{BQ}$ to meet the parallel to A^* and B^* ,



respectively. Then, by similarity,

$$\frac{\overrightarrow{AR}}{\overrightarrow{RB}} = \frac{\overrightarrow{A^{*}O}}{\overrightarrow{OB^{*}}} \;, \quad \frac{\overrightarrow{BP}}{\overrightarrow{PO}} = -\frac{\overrightarrow{AB}}{\overrightarrow{A^{*}O}} \;, \quad \frac{\overrightarrow{OQ}}{\overrightarrow{OA}} = -\frac{\overrightarrow{B^{*}O}}{\overrightarrow{BA}} \;.$$

From this, the result is immediate.

Conversely, if the product is 1 , draw BQ and AP and denote their intersection by C . Extend OC to its intersection R^{*} with AB . From the original proposition

$$\frac{\overrightarrow{AR^{*}}}{\overrightarrow{R^{*}B}} \; \frac{\overrightarrow{BP}}{\overrightarrow{PO}} \; \frac{\overrightarrow{OQ}}{\overrightarrow{OA}} = 1 \;,$$

hence,

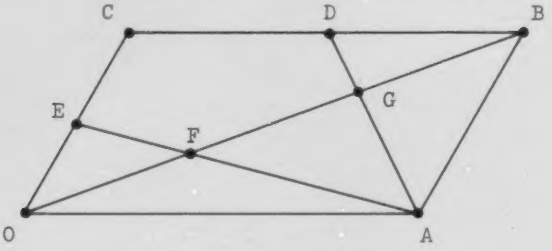
$$\frac{\overrightarrow{AR^{*}}}{\overrightarrow{R^{*}B}} = \frac{\overrightarrow{AR}}{\overrightarrow{RB}} \;,$$

which implies $R = R^{*}$. Hence the three lines AP , BQ , OR are collinear at C .

Both proofs of the converse require that AP and BQ intersect. If not then AP , BQ , and OR are parallel and a separate proof is needed.

13. Let (OABC) be a parallelogram, D the midpoint of $\overline{BC}$, and E the midpoint of $\overline{CO}$. Show that the lines AD and AE divide the diagonal $\overline{OB}$ in thirds.

Let F be the point where AE meets OB . Note that $\vec{E} = \frac{1}{2}\vec{C}$ and



$\vec{B} = \vec{A} + \vec{C}$. Consequently,

$$\begin{aligned}\vec{F} &= \mu\vec{B} = \mu(\vec{A} + \vec{C}) \\ &= (1 - \nu)\vec{A} + \nu\vec{E} = (1 - \nu)\vec{A} + \frac{\nu}{2}\vec{C} .\end{aligned}$$

Since $\vec{A}$ and $\vec{C}$ are noncollinear, the two representations of F a linear combination of $\vec{A}$ and $\vec{C}$ must be the same:

$$\mu = 1 - \nu = \frac{\nu}{2} .$$

It follows that $\mu = \frac{1}{3}$, or $\vec{F} = \frac{1}{3}\vec{B}$. In the same way it may be shown for the intersection G of AD and OB , that $\vec{G} = \frac{2}{3}\vec{B}$.

A14. Let $P_1 , P_2 , \ldots , P_n$ be the consecutive vertices of a regular polygon of n sides. Show that $\vec{P}_1 + \vec{P}_2 + \ldots + \vec{P}_n = \vec{O}$ where O is the center of the polygon. Is this result independent of the choice of origin?

Consider a rotation R about O through the angle $\frac{2\pi}{n}$. A rotation about O maps any figure into a congruent figure; hence $R(\vec{U} + \vec{V}) = R(\vec{U}) + R(\vec{V})$. Furthermore, in such a rotation through an angle between 0 and 2π , only the point O remains fixed. At the same time, $R : \vec{P}_i \longrightarrow \vec{P}_{i+1}$ (for $i = 1, 2, \ldots, n-1$) and $R : \vec{P}_n \longrightarrow \vec{P}_1$. Consequently,

$$\sum_{i=1}^{n} \vec{P}_i = \sum_{i=1}^{n} R(\vec{P}_i) = R(\sum_{i=1}^{n} \vec{P}_i)$$

i.e., the point corresponding to the sum of the position vectors is fixed. It follows that $\sum_{i=1}^{n} \vec{P}_i = \vec{0}$. Plainly, this result cannot hold for any origin other than the center.

15. Prove that the bisectors of the interior angles of a triangle are concurrent.

Let the vertices of the triangle be O , A , B . From the solution to Number 5, the angle bisectors from A and B are given by

$$\vec{X} = \vec{A} + p\left(\frac{\vec{B} - \vec{A}}{|\vec{A} - \vec{B}|} - \frac{\vec{A}}{|\vec{A}|}\right), \tag{1}$$

$$\vec{Y} = \vec{B} + q\left(\frac{\vec{A} - \vec{B}}{|\vec{A} - \vec{B}|} - \frac{\vec{B}}{|\vec{B}|}\right), \tag{2}$$

respectively. These intersect (X = Y) when

$$\frac{\vec{B} - \vec{A}}{|\vec{B} - \vec{A}|}(p + q) = \left(\frac{p}{|\vec{A}|} - 1\right)\vec{A} - \left(\frac{q}{|\vec{B}|} - 1\right)\vec{B} .$$

Since $\vec{A}$ and $\vec{B}$ are not collinear, this implies:

$$\frac{p + q}{|\vec{B} - \vec{A}|} = 1 - \frac{p}{|\vec{A}|} = 1 - \frac{q}{|\vec{B}|} ;$$

whence, $p = q\,\dfrac{|\vec{A}|}{|\vec{B}|}$ and

$$q = \frac{|\vec{B}| \cdot |\vec{B} - \vec{A}|}{|\vec{A}| + |\vec{B}| + |\vec{B} - \vec{A}|} .$$

Insert this in (2) to obtain the point of intersection

$$\frac{|\vec{A}| \cdot |\vec{B}|}{|\vec{A}| + |\vec{B}| + |\vec{B} - \vec{A}|}\left(\frac{\vec{A}}{|\vec{A}|} + \frac{\vec{B}}{|\vec{B}|}\right) ;$$

this lies on the angle bisector from O , according to the solution of Number 3, as we sought to prove.

Alternative Solution. Use Number 12. Let the points where the bisectors from O , A , B meet the opposite sides be $\vec{R}$, $\vec{P}$, $\vec{Q}$, respectively. From Number 3,

$$\vec{R} = a\left(\frac{\vec{A}}{|\vec{A}|} + \frac{\vec{B}}{|\vec{B}|}\right)$$

and from Number 9(a),

$$\frac{\overrightarrow{AR}}{\overrightarrow{RB}} = \frac{|\vec{A}|}{|\vec{B}|} .$$

Similarly, from (1) and (2),

$$\frac{\overrightarrow{BP}}{\overrightarrow{PO}} = \frac{|\vec{B} - \vec{A}|}{|\vec{A}|} , \quad \frac{\overrightarrow{OQ}}{\overrightarrow{QA}} = \frac{|\vec{B}|}{|\vec{B} - \vec{A}|} .$$

Thus, the product of the ratios is 1 , and by Ceva's Theorem, the angle bisectors are concurrent.

16. A median of a tetrahedron is a line segment joining any one vertex to the centroid of the other three. Show that the medians of the tetrahedron are concurrent at the centroid of its four vertices. Show also that the segment of median between tne centroid and vertex is $\frac{3}{4}$ of the total length of the median.

Let P_1 , P_2 , P_3 , P_4 be the vertices of the tetrahedron. Let the centroid of the face opposite P_1 be $C_1 = \frac{1}{3}(P_2 + P_3 + P_4)$, and the centroid of the tetrahedron, $M = \frac{1}{4}(P_1 + P_2 + P_3 + P_4)$. The equation of the median from P_1 is

$$\vec{X} = (1 - \lambda)\vec{P}_1 + \lambda\vec{C}_1 , \qquad 0 \leq \lambda \leq 1$$

and clearly, for $\lambda = \frac{3}{4}$,

$$\vec{M} = \frac{1}{4} P_1 + \frac{3}{4} C$$

so that M lies on the median. Since any vertex may be designated as P_1 , the proof is complete.

17. The segment joining the midpoint of any edge of a tetrahedron to the midpoint of the opposite edge is bisected by the centroid of the four vertices.

In the notation of Number 16 let $\overline{P_1P_2}$ be one edge of the tetrahedron, $\overline{P_3P_4}$ the opposite edge. The segment joining the midpoints of these edges is given by

$$\vec{X} = (1 - \lambda)\,\frac{\vec{P}_1 + \vec{P}_2}{2} + \lambda\,\frac{\vec{P}_3 + \vec{P}_4}{2}\ .$$

Clearly $X = M$ for $\lambda = \frac{1}{2}$, which proves the assertion.

18. The eight planes, each containing one edge and bisecting the opposite edge of a tetrahedron, are concurrent.

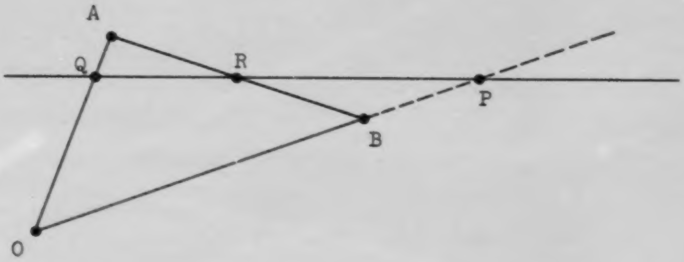
Such a plane contains the corresponding line segment of Number 17, hence, also the centroid of the tetrahedron.

19. Let $\vec{A}$, $\vec{B}$ be any noncollinear vectors, let $\mathcal{L}$ be any line in the plane OAB which contains no vertex of the triangle $\overline{OAB}$ and is parallel to no side. Let P be the intersection of $\mathcal{L}$ with OB, Q the intersection with OA, and R the intersection with AB. If α, β, γ are given by

$$\vec{R} - \vec{A} = \alpha(\vec{B} - \vec{R})\ ,\ \vec{B} - \vec{P} = \beta\vec{P}\ ,\ \vec{Q} = \gamma(\vec{Q} - \vec{A})$$

show that $\alpha\beta\gamma = -1$. Conversely, if P, Q, R are points satisfying $\alpha\beta\gamma = -1$ then they are collinear. (Menelaus's Theorem.)

In tne notation of Number 12, we have $\vec{P} = \mu\vec{B}$, $\vec{Q} = \nu\vec{A}$, $\vec{R} = (1 - \lambda)\vec{A} + \lambda\vec{B}$.



Now, from the collinearity of P, Q, and R (observe that $\vec{P} \neq \vec{0}$ since P and Q lie on different lines and not at the intersection),

$$\vec{R} = (1 - \tau)\vec{P} + \vec{Q} = \nu\tau\vec{A} + \mu(1 - \tau)\vec{B}\ .$$

Consequently, from the two representations of R,

$$\nu\tau = 1 - \lambda\ \ ,\ \ \mu(1 - \tau) = \lambda\ ,$$

whence,

$$(1) \qquad \tau = \frac{1 - \lambda}{\nu} = 1 - \frac{\lambda}{\mu}$$

Now, observe that,

$$(2) \qquad \lambda = \frac{\alpha}{1 + \alpha} \ , \quad \mu = \frac{1}{\beta + 1} \ , \quad \nu = \frac{\gamma}{1 + \gamma} \ .$$

Insert these expressions for λ , μ and ν in (1) to obtain the desired results: $\alpha\beta\gamma = 1$.

Conversely, if $\alpha\beta\gamma = -1$, let R^* be the point where PQ intersects AB . Then both $\vec{R} - \vec{A} = (B - \vec{R})$ and $\vec{R}^* - \vec{A} = (\vec{B} - \vec{R}^*)$ and $\vec{R}^*$ has the same representation in terms of $\vec{A}$ and $\vec{B}$ as $\vec{R}$; therefore, $\vec{R}^* = \vec{R}$.

TC11-4. Products of Two Vectors.

A binary relation which satisfies the linearity conditions given at the beginning of the section is called bilinear. The third type of invariant bilinear product is the dyadic product $\vec{A} > < \vec{B}$ which is the linear transformation represented by the matrix

$$\vec{A} > < \vec{B} = \begin{bmatrix} A_xB_x & A_xB_y & A_xB_z \\ A_yB_x & A_yB_y & A_yB_z \\ A_zB_x & A_zB_y & A_zB_z \end{bmatrix} .$$

The dyadic product also has useful applications.

The right-handed convention for the cross product refers to physical standards, the face of a clock, the fingers of a hand. It may seem primitive to use sign language or physical devices rather than purely mathematical methods to prescribe an orientation convention. The axioms of geometry, however, are unaffected by reflections in a plane. The two half-spaces on either side of a line have identical geometrical descriptions. Thus there are no preferred orientations in geometry and it is impossible to remain entirely within the abstract frame of mathematics and characterize a preferred side of a plane. In the same vein imagine the problem of conveying the distinction between right and left to someone who does not know our conventions of language (a child, say) without appealing to physical objects. We do communicate geometrical ideas with physical objects, pictures, and for the purposes of such communication, and for applications as well we adopt conventions by reference to standard objects.

Should the question be brought up it may be pointed out that failure of parity conservation in weak particle interactions is an entirely different issue. It was believed that the laws of physics had mirror symmetry, hence that the laws of physics define no preferred orientation. This would not preclude the choice of a standard reference object. It has now been found that mirror symmetry fails in the quantum domain. Thus it is possible to adopt an orientation convention without appealing to anything outside the laws of physics.

In a general change of coordinate frame the components of $\vec{A} \times \vec{B}$ do not transform like a vector but like the components of the matrix of the linear transformation with the skew-symmetric representation

$$M = \begin{bmatrix} 0 & -V_z & V_y \\ V_z & 0 & -V_x \\ -V_y & V_x & 0 \end{bmatrix} .$$

If the z-axis is reversed the matrix of the linear transformation becomes

$$M^* = \begin{bmatrix} 0 & -V_z & -V_y \\ V_z & 0 & V_x \\ V_y & -V_x & 0 \end{bmatrix} .$$

Thus $V_x^* = -V_x = -U_x^*$, $V_y^* = -V_y = -U_y^*$, $V_z^* = V_z = -U_z^*$ as we have seen in the text.

Solutions Exercises 11-4

1. Verify Properties (5a-d) of the dot product.

All four properties are direct algebraic consequences of (3). For an independent proof of (3) use the first solution of Number 2.
Alternative solution. (without the help of (3)).

Equation (5a) follows at once from (1) and the commutativity of ordinary multiplication.

Verify Equation (5b) by showing separately

(i) $$\vec{A} \cdot (\lambda\vec{B}) = \lambda(\vec{A} \cdot \vec{B})$$

and

(ii) $$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C} .$$

To prove (i) observe first that if $\lambda > 0$ the angle between $\vec{A}$ and $\lambda\vec{B}$ is still θ and

$$\vec{A}\cdot(\lambda\vec{B}) = |\vec{A}||\lambda\vec{B}| \cos\theta = \lambda|\vec{A}||\vec{B}| \cos\theta$$
$$= \lambda\vec{A}\cdot\vec{B} .$$

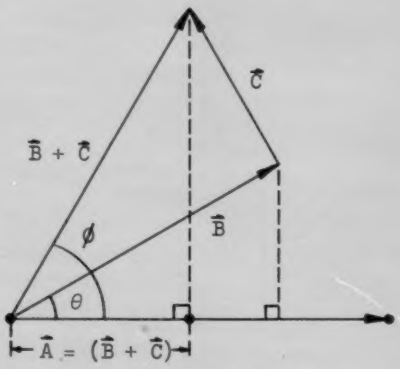
If $\lambda < 0$, then $\lambda\vec{B}$ has the direction opposite to $\vec{B}$ and the angle between $\vec{A}$ and $\lambda\vec{B}$ is $\pi - \theta$; hence

$$\vec{A}\cdot(\lambda\vec{B}) = |\vec{A}||\lambda\vec{B}| \cos(\pi-\theta) = (-\lambda)|A||B|(-\cos\theta)$$
$$= \lambda\vec{A}\cdot\vec{B} .$$

To prove (ii) observe that the projection of $\vec{B}+\vec{C}$ on the line OA is simply the sum of the projections of $\vec{B}$ and $\vec{C}$ on the line (see figure). Thus, if θ is the angle between $\vec{A}$ and $\vec{B}$, ϕ between $\vec{A}$ and $\vec{C}$, ψ between $\vec{A}$ and $\vec{B}+\vec{C}$, then

$$|\vec{B}+\vec{C}| \cos\psi = |\vec{B}| \cos\theta + |\vec{C}| \cos\phi .$$

Multiply by $|\vec{A}|$ to obtain (ii).



Equation (5c) follows from $\theta = 0$, $\cos\theta = 1$.

Inequality (5d) follows from $|\cos\theta| \leq 1$.

2. Obtain the coordinate representation (3) of the dot product.
As suggested in the text, use

$$\vec{A}\cdot\vec{B} = \frac{1}{2}(|\vec{A}|^2 + |\vec{B}|^2 - |\vec{B}-\vec{A}|^2)$$
$$= \frac{1}{2}((A_x{}^2 + A_y{}^2 + A_z{}^2) + (B_x{}^2 + B_y{}^2 + B_z{}^2) - ([B_x - A_x]^2 + [B_y - A_y]^2 + [B_z - A_z]^2).$$

Alternative Solution. Use the Properties (5) of inner product as derived independently of (3) in the alternative solution of Number 1.

For this purpose let $\vec{i}$, $\vec{j}$, and $\vec{k}$ be the unit vectors in the directions of the positive x-, y-, and z-axis, respectively. These unit vectors have the coordinate representations $\vec{i} = (1,0,0)$, $\vec{j} = (0,1,0)$, $\vec{k} = (0,0,1)$. Consequently we may represent $\vec{A}$ and $\vec{B}$ as linear combinations of these unit vectors with the components in the directions of the corresponding axes as coefficients:

$$\vec{A} = (A_x, A_y, A_z) = A_x\vec{i} + A_y\vec{j} + A_z\vec{k} ,$$

$$\vec{B} = (B_x, B_y, B_z) = B_x\vec{i} + B_y\vec{j} + B_z\vec{k} .$$

From (1), $\vec{i}\cdot\vec{i} = \vec{j}\cdot\vec{j} = \vec{k}\cdot\vec{k} = 1$ and $\vec{i}\cdot\vec{j} = \vec{j}\cdot\vec{k} = \vec{k}\cdot\vec{i} = 0$. Consequently, taking the dot product of the two linear combinations and using (5a) and (5b), we obtain

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z .$$

3. (a) Show that the perpendicular projection of the vector $\vec{B}$ on the line of $\vec{A}$ is the vector

$$\vec{B}_A = \frac{|\vec{B}| \cos\theta}{|\vec{A}|}\vec{A} = \frac{\vec{B}\cdot\vec{A}}{\vec{A}^2}\vec{A} .$$

The vector $\vec{B}_A$ is called the component of $\vec{B}$ in the direction of $\vec{A}$.

See Figure 11-4b.

(b) Write $\vec{B}$ in the form $\vec{B} = \vec{B}_A + \vec{B}^A$ and show that $\vec{B}^A$ is perpenperpendicular to $\vec{B}_A$. The vector $\vec{B}^A$ is called the component of $\vec{B}$ perpendicular to $\vec{A}$.

Observe, if $\vec{A} \neq \vec{0}$,

$$\vec{B}^A\cdot\vec{A} = (\vec{B} - \vec{B}_A)\cdot\vec{A} = (\vec{B} - \frac{\vec{B}\cdot\vec{A}}{A^2}\vec{A})\cdot\vec{A}$$

$$= \vec{B}\cdot\vec{A} - \vec{B}\cdot\vec{A}\,\frac{\vec{A}^2}{\vec{A}^2} = 0 .$$

The case $\vec{A} = \vec{0}$ is trivial.

4. Let $\vec{A}$ and $\vec{B}$ be noncollinear vectors. Minimize $|\vec{B} - \lambda\vec{A}|$ and interpret geometrically.

Equivalently, minimize

$$|\vec{B} - \lambda\vec{A}|^2 = (\vec{B} - \lambda\vec{A})\cdot(\vec{B} - \lambda\vec{A})$$

$$= \vec{B}^2 - 2\lambda(\vec{B}\cdot\vec{A}) + \lambda^2\vec{A}^2 .$$

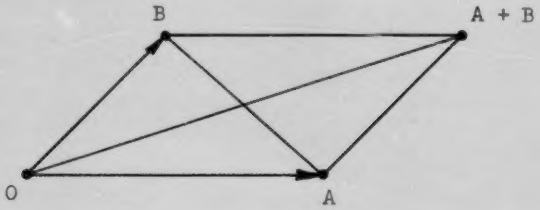
The minimum occurs at $\lambda = \frac{\vec{B}\cdot\vec{A}}{\vec{A}^2}$, i.e., when

$$\vec{B} - \lambda\vec{A} = \vec{B} - \frac{\vec{B}\cdot\vec{A}}{\vec{A}^2}\vec{A} .$$

Geometrically, the interpretation of this statement is that the nearest point on the line OA to B is the foot of the perpendicular from B upon OA .

5. Prove that the diagonals of a rhombus intersect at right angles.

Represent the rhombus by means of the parallelogram law for $\vec{A} + \vec{B}$

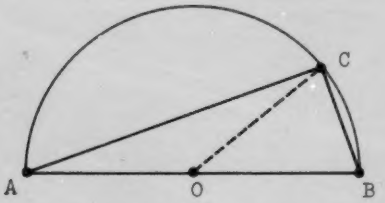


where $|\vec{A}| = |\vec{B}|$. The diagonals are represented by $\vec{B} + \vec{A}$ and $\vec{B} - \vec{A}$, and

$$(\vec{B} + \vec{A})\cdot(\vec{B} - \vec{A}) = \vec{B}^2 - \vec{A}^2 = 0 .$$

6. Prove that an angle inscribed in a semicircle is a right angle.

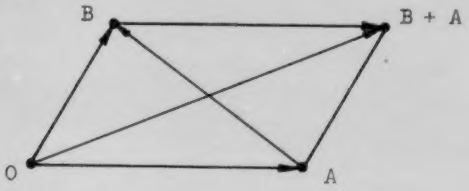
Use $|\vec{A}| = |\vec{B}| = |\vec{C}|$ and $\vec{A} = -\vec{B}$ to obtain



$$(\vec{A} - \vec{C})\cdot(\vec{B} - \vec{C}) = -\vec{A}^2 + \vec{C}^2 = 0 .$$

7. (a) Show that the sum of the squares of the sides of a parallelogram is equal to the sum of the squares of the diagonals.

Represent the parallelogram by means of a vector as indicated in the accompanying figure. The sum of the squares of the sides is

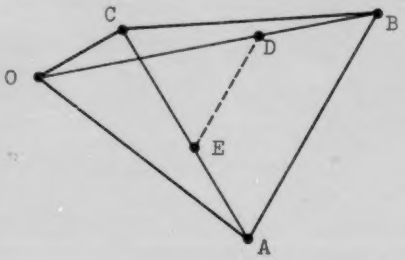


$2\vec{A}^2 + 2\vec{B}^2$, and the sum of the squares of the diagonals is

$$(\vec{B} - \vec{A})^2 + (\vec{B} + \vec{A})^2 = (\vec{B}^2 - 2\vec{B}\cdot\vec{A} + \vec{A}^2) + (\vec{B}^2 + 2\vec{B}\cdot\vec{A} + \vec{A}^2) .$$

(b) Show for an arbitrary quadrilateral that the sum of the squares of the sides exceeds the sum of the squares of the diagonals by four times the square of the distance between the midpoints of the diagonals.

Let the successive veetices of the quadrilateral be O , A , B , C , and let the midpoints of the diagonals $\overline{OB}$ and $\overline{AC}$ be D and E ,



respectively (see figure). Note that $\vec{D} = \frac{1}{2}\vec{B}$ and $\vec{E} = \frac{1}{2}(\vec{A} + \vec{C})$. For the sum of the squares of the sides, obtain

$$\begin{aligned} s &= \vec{A}^2 + (\vec{B} - \vec{A})^2 + (\vec{C} - \vec{B})^2 + \vec{C}^2 \\ &= 2\vec{A}^2 + 2\vec{B}^2 + 2\vec{C}^2 - 2\vec{B}\cdot(\vec{A} + \vec{C}) , \end{aligned}$$

for the sum of the squares of the diagonals, $\alpha = \vec{B}^2 + (\vec{C} - \vec{A})^2 = \vec{A}^2 + \vec{B}^2 + \vec{C}^2 - 2\vec{A}\cdot\vec{C}$, and for the square of distance between D and E , $m = \frac{1}{4}(\vec{A} + \vec{C} - \vec{B})^2 = \frac{1}{4}[\vec{A}^2 + \vec{B}^2 + \vec{C}^2 + 2\vec{A}\cdot\vec{C} - 2\vec{B}\cdot(\vec{A}+\vec{C})]$. Clearly, $s = d + 4m$, as stated.

8. Given $\vec{U} = (1,1,1)$, find vectors $\vec{V}$, $\vec{W}$, so that $(\vec{U},\vec{V},\vec{W})$ is a right-handed triple of mutually perpendicular vectors.

Take any vector which is not collinear with $\vec{U}$, say $\vec{I} = (1,0,0)$. Set $\vec{V} = \vec{I} \times \vec{U} = (0,-1,1)$, $\vec{W} = \vec{U} \times \vec{V} = (2,-1,-1)$.
(The use of a right-handed coordinate frame is taken for granted here.)

9. In connection with the definition of cross product, why must a function, continuous on an interval, which can take on only the values ± 1 be constant?

If the function took on both values it would have to take on all values between as well, (Intermediate Value Theorem).

10. Show that $\vec{A} \times \vec{B} = \vec{A} \times \vec{B}^A$, (see Number 3).

From Number 3

$$\begin{aligned} \vec{A} \times \vec{B} &= \vec{A} \times (\vec{B}_A + \vec{B}^A) \\ &= \vec{A} \times \vec{B}_A + A \times \vec{B}^A \\ &= \frac{(\vec{B} \cdot \vec{A})}{A^2}(\vec{A} \times \vec{A}) + \vec{A} \times \vec{B}^A \\ &= \vec{0} + \vec{A} \times \vec{B}^A . \end{aligned}$$

11. (a) Express the vector $\vec{V} = (\vec{A}^C)^B - (\vec{A}^B)^C$ in terms of dot and cross products.

From Number 3, if neither $\vec{B}$ nor $\vec{C}$ is $\vec{0}$,

$$\vec{A}^C = \vec{A} - \frac{\vec{A} \cdot \vec{C}}{\vec{C}^2} \vec{C}$$

and

$$\begin{aligned} (\vec{A}^C)^B &= \vec{A}^C - \frac{\vec{A}^C \cdot \vec{B}}{\vec{B}^2} \vec{B} \\ &= \vec{A} - \frac{\vec{A} \cdot \vec{C}}{\vec{C}^2} \vec{C} - \frac{\vec{A} \cdot \vec{B}}{\vec{B}^2} \vec{B} + \frac{(\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{C})\vec{B}}{\vec{B}^2\vec{C}^2} . \end{aligned}$$

Interchange C and B in this last result and subtract to obtain

$$\vec{V} = \frac{\vec{B} \cdot \vec{C}}{\vec{B}^2\vec{C}^2}[(\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}] .$$

Note that the expression in brackets is the triple cross product of Example 11-4e, so that

$$\vec{V} = \frac{\vec{B} \cdot \vec{C}}{\vec{B}^2\vec{C}^2}[\vec{A} \times (\vec{B} \times \vec{C})] \ .$$

(b) Compare $\vec{V}$ with $\vec{U} = (\vec{A}_C)_B - (\vec{A}_B)_C$ and $\vec{W} = (\vec{A}_B)^C - (\vec{A}^C)_B$.

$\vec{V} = \vec{U} = \vec{W}$.

12. In the text, given noncollinear vectors $\vec{A}$ and $\vec{B}$ we chose a right-handed fundamental set $\{\vec{i}_0, \vec{j}_0, \vec{k}_0\}$ with $\vec{i}_0$ in the direction of $\vec{A}$, with $\vec{j}_0$ in the plane of $\vec{A}$ and $\vec{B}$ so that the rotation in the plane from $\vec{A}$ to $\vec{j}_0$ is in the same sense as that from $\vec{A}$ to $\vec{B}$, and with $\vec{k}_0$ perpendicular to the plane of $\vec{A}$ and $\vec{B}$. Express $\vec{i}_0$, $\vec{j}_0$, $\vec{k}_0$ in terms of $\vec{A}$ and $\vec{B}$.

$$\vec{i}_0 = \frac{1}{|\vec{A}|}\vec{A}$$

$$\vec{j}_0 = \frac{\vec{B}^A}{|\vec{B}^A|} = \frac{(\vec{A}^2)\vec{B} - (\vec{A} \cdot \vec{B})\vec{A}}{|(\vec{A}^2)\vec{B} - (\vec{A} - \vec{B})A|}$$

$$\vec{k}_0 = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|} \ .$$

13. Verify (16) for the degenerate cases ignored in its derivation.

In the derivation of (16) it was assumed that B and C are noncollinear. Here we assume B and C are collinear, say $C = \lambda B$. Then

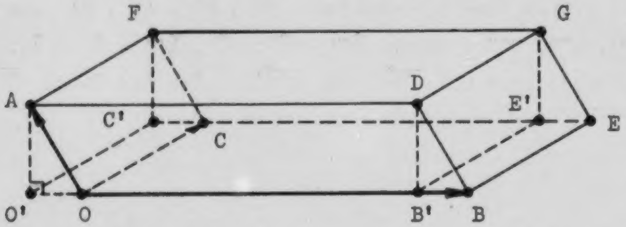
$$\vec{A} \times (\vec{B} \times \vec{C}) = \lambda\vec{A} \times (\vec{B} \times \vec{B}) = \vec{0}$$

and

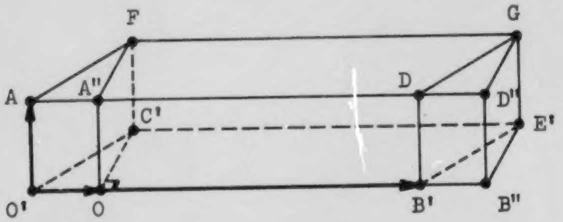
$$(\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C} = \lambda[(\vec{A} \cdot \vec{B})\vec{B} - (\vec{A} - \vec{B})\vec{B}] = \vec{0} \ .$$

14. Prove the result of Example 11-4f.

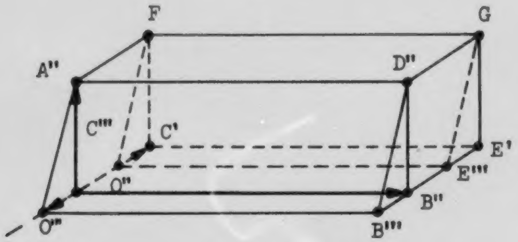
The proof is similar in method to the derivation of the formula for



the area of a parallelogram by constructing an equivalent rectangle. In this case, construct an equivalent rectangular parallelepiped. The three



figures show how this may be done. Let $\{\vec{A},\vec{B},\vec{C}\}$ be a right-handed set. The parallelepiped with initial edges $\overline{OA}$, $\overline{OB}$, $\overline{OC}$ is indicated in the first figure by the vertices O , A , D , B , C , F , G , E where $\vec{D} = \vec{A} + \vec{B}$, $\vec{E} = \vec{B} + \vec{C}$, $\vec{F} = \vec{C} + \vec{A}$, $\vec{G} = \vec{A} + \vec{B} + \vec{C}$. Next, introduce new points by $\vec{O'} = \vec{A}_B$, $\vec{B'} = \vec{B} + \vec{A}_B$, $\vec{C'} = \vec{C} + \vec{A}_B$ and $E' = \vec{E} + \vec{A}_B$. This defines a new parallelepiped with the initial edges $\overline{O'A}$, $\overline{O'B'}$, $\overline{O'C'}$, where $\overline{O'A}$ is perpendicular to $\overline{O'B}$. Thus, the two faces $\overline{AO'B'D}$ and $\overline{FC'E'G}$ are rectangular. The two parallelepipeds have equal volume since the prisms $\overline{O'OAC'CF}$ and $\overline{B'BDE'EF}$ are congruent (the second is obtained from the first by the translation $\vec{B}$) . In the second figure the same idea is repeated with $\vec{C}_B$, the projection of $\overrightarrow{O'C'}$ on $\overrightarrow{O'B'}$ and the vertices O' , A , B' , D are replaced by O'' , A'' , B'' , D'' , where $\vec{O''} = \vec{O'} + \vec{C}_B$, etc. The new parallelepiped with initial edges $\overline{O''A''}$, $\overline{O''B''}$, $\overline{O''C'}$ has the rectangular faces $\overline{O''A''D''B''}$, $\overline{O''B''E'C'}$



(and the faces opposite). In the third figure, the process is repeated once again; this time the vertices O'', B'', E', C' are replaced by O''', B''', E''', C''', where $\vec{O}''' = \vec{O}'' + \vec{U}$, $\vec{B}''' = \vec{B}'' + \vec{U}$. etc.. $\vec{U}$ being the projection of $\overrightarrow{O''A''}$ on $\overrightarrow{O''C'}$. Now the faces $\overline{O'''B'''E'''C'''}$ and $\overline{O'''C'''FA''}$ are rectangles and we must verify that the third face at O''' is a rectangle, namely, that $(\vec{A}'' - \vec{O}''') \cdot (\vec{B}''' - \vec{O}''') = 0$. For this, observe that $\vec{B}''' - \vec{O}''' = \vec{B}'' - \vec{O}'' = \vec{B}$ is perpendicular to both $\vec{A}'' - \vec{O}''$ and to $\vec{U}$ (which is parallel to $\overrightarrow{O''C'}$). Since $\vec{A}'' - \vec{O}''' = (\vec{A}'' - \vec{O}'') - \vec{U}$ it follows that $\vec{B}$ is perpendicular to this vector. Thus, at this stage we have obtained a rectangular parallelepiped of the same volume as the initial parallelepiped.

The (signed) volume of a rectangular parallelepiped does obey (17) since the three vectors are mutually perpendicular. In particular,

$$\begin{aligned} V &= \pm|O'''A''|\,|O'''B'''|\,|O'''C'''| \\ &= (\vec{A}'' - \vec{O}''') \cdot [(\vec{B}''' - \vec{O}''') \times (\vec{C}''' - \vec{O}''')] , \end{aligned}$$

where the sign is determined by the "handedness" of the vector triple. Now, insert the expression for $\vec{A}'' - \vec{O}'''$ above and observe, since $\vec{U}$ is parallel to $\vec{C}''' - \vec{O}'''$ that it is perpendicular to the cross product; hence,

$$\begin{aligned} V &= (\vec{A}'' - \vec{O}'') \cdot [(\vec{B}''' - \vec{O}'') \times (\vec{C}' - \vec{O}''')] \\ &= (\vec{A}'' - \vec{O}'') \cdot [\vec{B} \times (\vec{C}' - \vec{O}'')] \\ &= (\vec{A} - \vec{C}_B) \cdot [\vec{B} \times (\vec{C}' - \vec{C}_B)] \end{aligned}$$

where $\vec{C}_B$ is parallel to $\vec{B}$; hence,

$$V = \vec{A} \cdot [\vec{B} \times \vec{C}'] = \vec{A} \cdot [\vec{B} \times (\vec{C} + \vec{A}_B)] ;$$

where $\vec{A}_B$ is parallel to $\vec{B}$; hence,

$$V = \vec{A} \cdot [\vec{B} \times \vec{C}] .$$

15. Show $\vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C}$

$$\begin{aligned} &= -\vec{A} \cdot (\vec{C} \times \vec{B}) \\ &= \vec{B} \cdot (\vec{C} \times \vec{A}) \\ &= -\vec{B} \cdot (\vec{A} \times \vec{C}) \\ &= \vec{C} \cdot (\vec{A} \times \vec{B}) \\ &= -\vec{C} \cdot (\vec{B} \times \vec{A}) \end{aligned}$$

Can you give a general rule for the sign?

Apart from sign each of these triple scalar products is the volume of the parallelepiped with adjacent edges $\overline{OA}$, $\overline{OB}$, $\overline{OC}$. The sign of the triple scalar product depends on the handedness of the vectors. The handedness is reversed by an interchange of any two of the vectors, and is preserved by a cyclic permutation.

16. Let $\vec{A} + \vec{B} + \vec{C} = \vec{0}$. Show that

$$\vec{A} \times \vec{B} = \vec{B} \times \vec{C} = \vec{C} \times \vec{A} .$$

Interpret this result geometrically to obtain the law of sines for triangles.

Set $\vec{C} = -(\vec{A} + \vec{B})$ to obtain

$$\vec{B} \times \vec{C} = -\vec{B} \times (\vec{A} + \vec{B}) = -\vec{B} \times \vec{A} = \vec{A} \times \vec{B}$$

and

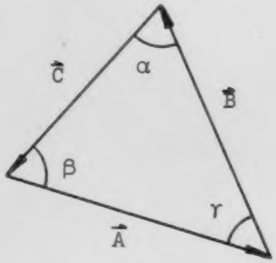
$$\vec{C} \times \vec{A} = -(\vec{A} + \vec{B}) \times \vec{A} = -\vec{B} \times \vec{A} = \vec{A} \times \vec{B} .$$

Since the sum of the vectors is 0 the sum represents a closed polygon, in this case a triangle. If α , β , γ are the vertex angles of the triangle as shown in the figure, then we have found

$$|\vec{A}|\,|\vec{B}|\sin\gamma = |\vec{B}|\,|\vec{C}|\sin\alpha = |\vec{C}|\,|\vec{A}|\sin\beta ;$$

hence,

$$\frac{\sin\alpha}{|\vec{A}|} = \frac{\sin\beta}{|\vec{B}|} = \frac{\sin\gamma}{|\vec{C}|} .$$



17. Use (16) and Number 15 to express

$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D})$$

in terms of dot products alone.

From Number 15, and from (16), successively,

$$\begin{aligned}(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) &= \vec{D} \cdot [(\vec{A} \times \vec{B}) \times \vec{C}] \\ &= \vec{D} \cdot [\vec{C} \times (\vec{B} \times \vec{A})] \\ &= (\vec{A} \cdot \vec{C})(\vec{D} \cdot \vec{B}) - (\vec{B} \cdot \vec{C})(\vec{D} \cdot \vec{A}) .\end{aligned}$$

Λ18. What is the shortest distance between the straight lines

$$\mathcal{L}_1 : \vec{X} = \vec{C} + s\vec{A}$$

$$\mathcal{L}_2 : \vec{Y} = \vec{D} + t\vec{B} \ ?$$

What is the equation of the line perpendicular to both?

From the solution to Number 4, since that solution is independent of the choice of origin, the shortest distance from a point to a line is that along a perpendicular. It follows that the shortest distance from $\mathcal{L}_1$ to $\mathcal{L}_2$ is along a segment perpendicular to both. If $\overline{XY}$ is this segment, then

(i) $$\vec{Y} - \vec{X} = \lambda(\vec{B} \times \vec{A})$$

(unless $\vec{A}$ and $\vec{B}$ are collinear, in which case the lines are parallel and any mutual perpendicular yields to the shortest distance). Insert the given expressions for $\vec{X}$ and $\vec{Y}$ into (i) to obtain

(ii) $$s\vec{A} - t\vec{B} + \vec{C} - \vec{D} = \lambda(\vec{A} \times \vec{B}) \ .$$

To eliminate s and t in (ii) take the dot product with $\vec{A} \times \vec{B}$ to obtain

$$\lambda = \frac{(\vec{C} - \vec{D}) \cdot (\vec{A} \times \vec{B})}{|\vec{A} \times \vec{B}|^2}$$

which gives the desired length.

To determine the equation of the mutual perpendicular, given its direction, only one point on the line is needed. Eliminate t and λ in (ii) to determine s and the point where the mutual perpendicular meets $\mathcal{L}_1$. For this, take the dot product with $B \times (\vec{A} \times \vec{B})$ in (ii) to obtain s by

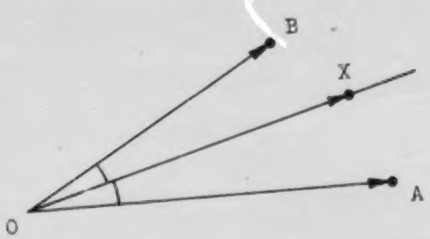
(iii) $$s\vec{A} \cdot [\vec{B} \times (\vec{A} \times \vec{B})] = (\vec{C} - \vec{D}) \cdot [\vec{B} \times (\vec{A} \times \vec{B})] \ .$$

This can be simplified by using the expression (16) for the triple cross product. However, for the present, it is sufficient to observe that the mutual perpendicular is given by

$$\vec{R} = \vec{C} + s\vec{A} + r(\vec{A} \times \vec{B})$$

where r is the parameter and s is fixed by (iii).

19. Use cross products to find the equation of the ray which bisects the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$. (Compare Exercises 11-3, No. 5.)



Let X be any point on the angle bisector. Since A and B lie on opposite sides of $\overrightarrow{OX}$, and the angles ∠AOX, ∠XOB are equal, we have

$$\frac{\vec{A} \times \vec{X}}{|\vec{A}|} = \frac{\vec{X} \times \vec{B}}{|\vec{B}|} .$$

To obtain $\vec{X}$ as a linear combination of $\vec{A}$ and $\vec{B}$, set $X = \lambda A + \mu B$ above to find

$$\mu \frac{\vec{A} \times \vec{B}}{|\vec{A}|} = \lambda \frac{\vec{A} \times \vec{B}}{|\vec{B}|} .$$

Thus, for $a = \mu|\vec{B}| = \lambda|\vec{A}|$, the result of Number 5 is obtained again:

$$\vec{X} = a\left(\frac{\vec{A}}{|\vec{A}|} + \frac{\vec{B}}{|\vec{B}|}\right).$$

20. Prove that

$$\vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) + \vec{C} \times (\vec{A} \times \vec{B}) = \vec{0} .$$

Expand out by (16).

21. Use (16) to find two different representations of $(\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D})$ and so establish an identity of the form $a\vec{A} + b\vec{B} + c\vec{C} + d\vec{D} = 0$. Hence, show how to express any vector as a linear combination of any three vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ for which $\vec{A} \cdot (\vec{B} \times \vec{C}) \neq \vec{0}$. (Compare Exercises 11-3, No. 8.)

$$\begin{aligned}(\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D}) &= [\vec{A} \cdot (\vec{C} \times \vec{D})]\vec{B} - [\vec{B} \cdot (\vec{C} \times \vec{D})]\vec{A} \\ &= [\vec{D} \cdot (\vec{A} \times \vec{B})]\vec{C} - [(\vec{A} \times \vec{B}) \cdot \vec{C}]\vec{D} .\end{aligned}$$

Since the coefficient of $\vec{D}$ is not 0 (see Number 15) the result is established.

22. Solve $\vec{X} = \vec{A} + (\vec{B} \times \vec{X})$.

If $\vec{A}$ and $\vec{B}$ are noncollinear then $\vec{A}$, $\vec{B}$ and $\vec{A} \times \vec{B}$ are noncoplanar. Set

$$\vec{X} = a\vec{A} + b\vec{B} + c\vec{A} + \vec{B} .$$

Take the cross product with $\vec{B}$ and use the given equation to obtain

$$\begin{aligned} \vec{B} \times \vec{X} = \vec{X} - \vec{A} &= (a - 1)\vec{A} + b\vec{B} + c\vec{A} \times \vec{B} \\ &= -a(\vec{A} \times \vec{B}) + c[\vec{B}^2 \vec{A} - (\vec{A} \cdot \vec{B})\vec{B}] \end{aligned}$$

where the bracketed expression is the expansion of $B \times (A \times B)$ by (17). Equate coefficients to obtain

$$a - 1 = c\vec{B}^2 \quad , \quad b = -c(\vec{A} \cdot \vec{B}) \quad , \quad c = -a \quad ,$$

whence

$$a = \frac{1}{1 + \vec{B}^2} \quad , \quad b = \frac{\vec{A} \cdot \vec{B}}{1 + \vec{B}^2} \quad , \quad c = \frac{1}{1 + \vec{B}^2} \quad .$$

If $\vec{A}$ and $\vec{B}$ are collinear, and $\vec{B} \neq \vec{0}$, ($\vec{B} = \vec{0}$ is trivial), then $\vec{A} = \lambda\vec{B}$ and

$$\vec{X} = \lambda\vec{B} + (\vec{B} \times \vec{X}) .$$

Take the dot product with $\vec{B} \times \vec{X}$ to obtain $(\vec{B} \times \vec{X})^2 = 0$; hence, $\vec{B} \times \vec{X} = 0$. Thus $\vec{B}$ and $\vec{X}$ are collinear, and $\vec{X} = \lambda\vec{B} = \vec{A}$.

TC11-5. Vector Calculus and Curves.

The choice of spherical neighborhoods in the definition of limit for a vector function is the most "natural" one since the definition is then independent of the coordinate frame. Given a coordinate frame, instead of the given characterization of ϵ-neighborhood, we could adopt, say,

$$\operatorname{Max}\{|R_x - A_x| \ , \ |R_y - A_y| \ , \ |R_z - A_z|\} < \epsilon .$$

The two characterizations are equivalent for the purposes of analysis, limits do not depend upon which definition of ϵ-neighborhood is used (compare TC p. 475), but the second definition suffers from the inconvenience that it depends on the coordinate system.

The Solution (23) to the differential equation (22) of Example 11-5i is actually the equation of a ray, not an entire straight line, since

$$\omega(s) = \exp\{-\int_0^A \frac{d\sigma}{\phi(\sigma)}\} > 0 .$$

This comes about because the equation is singular when $\phi(s) = 0$, or $\vec{X} = \vec{A}$. The solution is actually the ray with initial point A in the direction of $\vec{X}_0 - \vec{A}$.

Solutions Exerciese 11-5

1. Prove that Definition 11-5 and Equation (3) are equivalent definitions of limit for a vector function.

   From Definition 3-2, Equation (3) holds if and only if for every positive ϵ there exists a δ such that $|\vec{F}(t) - \vec{A}| < \epsilon$ whenever $0 < |t - t_0| < \delta$.

2. (a) Prove Properties (4) and (5) for the limits of vector functions.

   Except for the distinction between vector and scalar function, the proofs are virtually identical to the proofs of the corresponding limit theorems for scalar functions. Note in both cases, as in Chapter 3, that the existence of the limits on the right implies the existence of the limit on the left.

   (b) Use the results of Part (a) to prove Property (6).

   Let $[\vec{i},\vec{j},\vec{k}]$ be the fundamental set of coordinate vectors, then

   $$\vec{v}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k} \ ;$$

   hence, by (4),

   $$\lim_{t \sim t_0} \vec{v}(t) = \lim_{t \sim t_0} [f(t)\vec{i}] + \lim_{t \sim t_0} [g(t)\vec{j}] + \lim_{t \sim t_0} [h(t)\vec{k}] \ .$$

   Now from (5), and $\lim\limits_{t \sim t_0} \vec{A} = \vec{A}$ for any constant vector $\vec{A}$,

   $$\lim_{t \sim t_0} f(t)\vec{i} = \left[\lim_{t \sim t_0} f(t)\right]\left[\lim_{t \sim t_0} \vec{i}\right]$$

   $$= \left[\lim_{t \sim t_0} f(t)\right] i \ ,$$

   with similar results for the other terms of the sum. Thus if

$$\lim_{t \to t_0} f(t) = a\ , \ \lim_{t \to t_0} g(t) = b\ , \ \lim_{t \to t_0} h(t) = c\ ,$$

then $\lim_{t \to t_0} \vec{v}(t) = \vec{A}$, where $\vec{A} = (a,b,c)$.

To complete the proof it is necessary to show that the existence of the limit of $\vec{v}(t)$ implies the existence of the limits of the component functions. It is only necessary to observe that if

$$|\vec{v}(t) - A| = \sqrt{[f(t) - a]^2 + [g(t) - a]^2 + [h(t) - a]^2} < \epsilon$$

then

$$|f(t) - a| < \sqrt{[f(t) - a]^2 + [g(t) - a]^2 + [h(t) - a]} < \epsilon\ .$$

(c) Prove Properties (7) and (8).

Apply the result of Part (b) to the coordinate representation of $\vec{u}(t) \cdot \vec{v}(t)$ and $\vec{u}(t) \times \vec{v}(t)$.

3. Prove the vector differentiation formulas (10) - (14).

For the proof of Formula (10) note that

$$\frac{\vec{v}(t) - \vec{v}(t_0)}{t - t_0} = \left(\frac{f(t) - f(t_0)}{t - t_0}\ , \ \frac{g(t) - g(t_0)}{t - t_0}\ , \ \frac{h(t) - h(t_0)}{t - t_0}\right)$$

and pass to the limit as t approaches t_0 with the aid of (6).

For the proofs of (11) - (13), apply (10) to the coordinate representations, and use the scalar formula for the derivative of a product. Similarly, for the proof of (14) apply the scalar chain rule to the coordinate representation of the composition.

4. Consider the function $\phi : t \longrightarrow |\vec{v}(t)|$ where $\vec{v}$ is differentiable. Differentiate

(a) $\phi(t)$.

(b) $\frac{1}{\phi(t)}$.

(c) $\phi(t)^2$.

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Set $\phi(t) = \sqrt{\vec{r}(t) \cdot \vec{r}(t)}$.

(a) $\phi(t) = \dfrac{\vec{r}\,'(t) \cdot \vec{r}(t)}{|\vec{r}(t)|}$.

(b) $-\dfrac{\phi'(t)}{\phi(t)^2} = -\dfrac{\vec{r}\,'(t) \cdot \vec{r}(t)}{|\vec{r}(t)|^3}$.

(c) $D_t[\vec{r}(t) \cdot \vec{r}(t)] = 2\vec{r}\,'(t) \cdot \vec{r}(t)$.

5. Obtain the formula for the arclength of a plane curve given in polar coordinates by $\rho = f(\theta)$.

Use θ as a parameter and set

$$x = \rho \cos \theta \ , \ y = \rho \sin \theta \ .$$

From $\vec{r}\,'(\theta) = (\frac{d\rho}{d\theta} \cos \theta - \rho \sin \theta \ , \ \frac{d\rho}{d\theta} \sin \theta + \rho \cos \theta)$ obtain

$$s = \int_{\rho_0}^{\rho} |\vec{r}\,'(\theta)|d\theta = \int_{\theta_0}^{\theta} \sqrt{\rho^2 + (\frac{d\rho}{d\theta})^2} \, d\theta \ .$$

If ρ is taken as the parameter, then

$$\vec{r}\,'(\rho) = (\cos \theta - \rho\theta' \sin \theta \ , \ \sin \theta + \rho\theta' \cos \theta)$$

and

$$s = \int_{\theta_0}^{\theta} \sqrt{1 + \rho^2(\frac{d\theta}{d\rho})^2} \, d\rho \ .$$

6. Sketch each of the following curves, give the points at which the tangent vector doesn't exist, and represent the curve in terms of arclength where possible. (If no z-coordinate is given, restrict the locus to the x, y-plane).

(a) $\begin{cases} x = a + b \sin t \\ y = c + d \cos t \ . \end{cases} \qquad (0 \leq t \leq 2\pi)$.

There are several cases. (i) If b and d are both non-zero, $b \neq d$, the curve is the ellipse

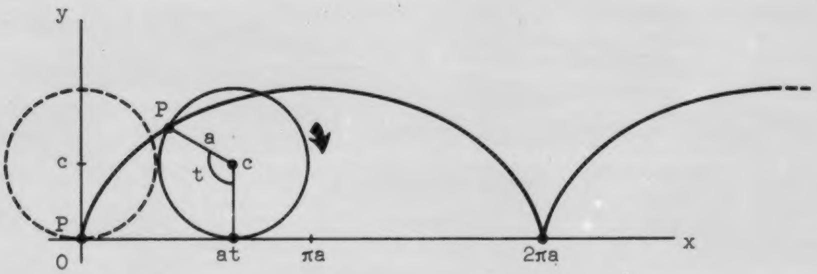
$$(\frac{x - a}{b})^2 + (\frac{y - c}{d})^2 = 1$$

with center at (a,c) and semi-axes $|b|$ and $|d|$. The tangent always exists. The arclength s is given by a so-called elliptic integral and is not an elementary function of t ; consequently x and y cannot be expressed as elementary functions of s . (ii) If $b = d \neq 0$ the curve is a circle of radius $|b|$. The tangent always exists. Since $|\vec{r}\,'(t)| = |b|$, the arclength is given by $s = |b|(t - t_0)$. (iii) If just one of b and d is non-zero, then the curve is a segment; if $b \neq 0$, $d = 0$ it is $\{(x,y) : y = c \ , \ a - |b| \leq x \leq a + |b|\}$. The tangent fails to exist at $t = n\pi$, (n integral); i.e., at the end points of the segment. (iv) If $b = d = 0$, the locus is the single point (a,c) . Since $\vec{r}\,'(t) = 0$ for all t , no tangent is defined, which is as it should be. Note that the restriction that the zeros of $\vec{r}\,'$ be isolated eliminates this singular case.

(b) The cycloid,

$$\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases} \qquad (a > 0 \ , \ 0 \leq t \leq 2\pi) .$$

Show that this curve is the locus of a point on a circle that rolls on a straight line without slipping.



Since $\vec{r}\,'(t) = a(1 - \cos t \ , \ \sin t)$, the tangent fails to exist when $t = 2n\pi a$ (n integral), where $\vec{r}\,'(t) = 0$. At these points the left-sided tangent is $(0,-1)$, and the right-sided tangent $(0,1)$, thus the points are cusps. From

$$|\vec{r}\,'(t)| = a\sqrt{2(1 - \cos t)} = 2a\left|\sin \frac{t}{2}\right| ,$$

It follows that

$$s = 2a\int_0^t \sin \frac{\tau}{2}\, d\tau = 4a\left(1 - \cos \frac{t}{2}\right) .$$

Consequently,

$$\begin{aligned} t &= 2 \arccos \left(1 - \frac{s}{4a}\right) \\ &= \arccos \left[2\left(1 - \frac{s}{4a}\right)^2 - 1\right] \\ &= \arcsin \left[\frac{1}{a}\left(1 - \frac{s}{4a}\right)\sqrt{8as - s^2}\right] , \end{aligned}$$

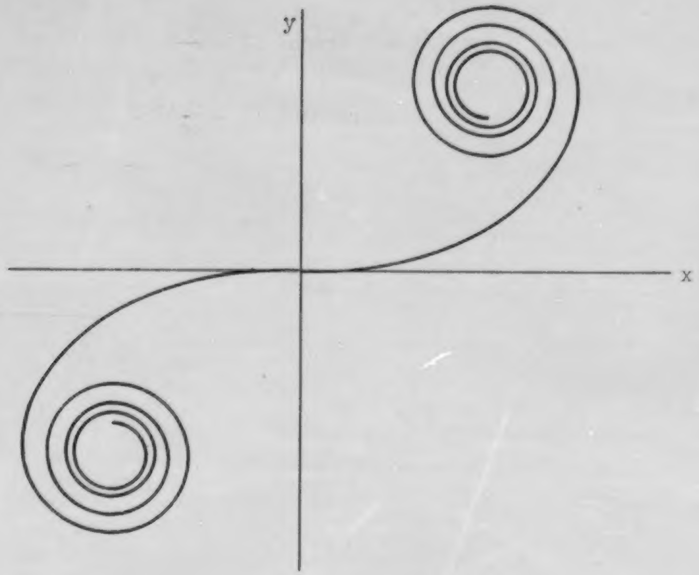
where $0 \leq s \leq 8a$. The parametric equations for the cycloid in terms of arclength are then

$$\begin{cases} x = 2a \arccos \left(1 - \frac{s}{4a}\right) - \left(1 - \frac{s}{4a}\right)\sqrt{8as - s^2} \\ y = \frac{1}{8a} (8as - s^2) . \end{cases}$$

To demonstrate the geometrical characterization of the cycloid consider the path traced out by a point P on a circle of radius a which rolls along the x-axis. Let C be the center of the circle and locate P with respect to the center. The center moves in straight line motion along the line $y = a$. As P rotates through the angle t with respect to the center, the length ta along the circle is rolled out on the x-axis. Consequently, if P is initially at the origin; after the rotation t with respect to C , we have $\vec{C} = (at,a)$ and $\vec{P} - \vec{C} = -a(\cos t , \sin t)$, which yields the given parametric representation immediately.

(c) The Cornu spiral,

$$\begin{cases} x = \int_0^t \cos u^2 \, du \\ y = \int_0^t \sin u^2 \, du . \end{cases}$$



This fascinating curve plays a role in the theory of diffraction of light (see Chapter 15). It has been much studied, but here the available tools permit a description of only the grosser features. Since $r'(t) = (\cos t^2, \sin t^2)$, we have $|r'(t)| = 1$ for all t, so that t is the arclength parameter, and the tangent is defined for all t. Since $\vec{r}(-t) = -\vec{r}(t)$, symmetry with respect to the origin obtains, so we need consider only positive t. Observe now that y has maxima at $t = \sqrt{(2n-1)\pi}$, minima at $t = \sqrt{2n\pi}$; furthermore it is easy to see that the maxima are decreasing and the minima increasing. For example, the difference between two successive minima of y is

$$\Delta = \int_{\sqrt{2n\pi}}^{\sqrt{(2n+2)\pi}} \sin u^2 \, du = \frac{1}{2} \int_{2n\pi}^{(2n+2)\pi} \frac{\sin v}{\sqrt{v}} \, dv$$

where the substitution $v = u^2$ is employed. Now,

$$\Delta = \frac{1}{2}\int_{2n\pi}^{(2n+1)\pi} \frac{\sin v}{\sqrt{v}} \, dv + \frac{1}{2}\int_{(2n+1)\pi}^{(2n+2)\pi} \frac{\sin v}{\sqrt{v}} \, dv$$

$$= \frac{1}{2}\left[\int_0^\pi \frac{\sin w}{\sqrt{w + 2n\pi}} \, dw - \int_0^\pi \frac{\sin w}{\sqrt{w + (2n+1)\pi}} \, dw\right]$$

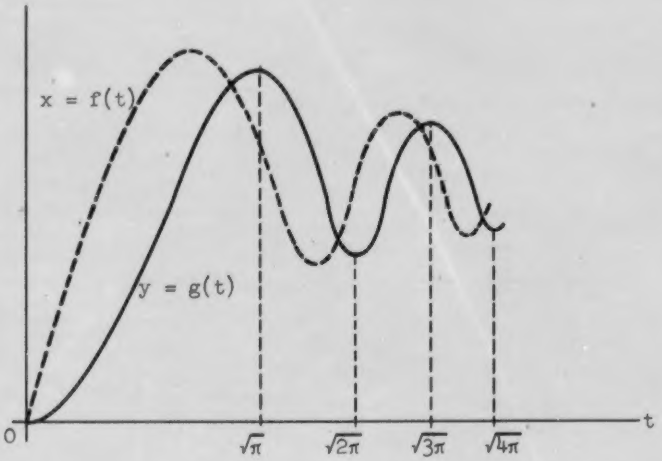
where, in the first integral we have put $v = w + 2n\pi$, in the second, $w = v + (2n + 1)\pi$. Hence,

$$\Delta = \frac{1}{2}\int_0^{\pi}\left[\frac{1}{\sqrt{w + 2n\pi}} - \frac{1}{\sqrt{w + (2n + 1)\pi}}\right] \sin w \, dw > 0 .$$

It is also clear for the difference between a minimum and the next maximum that

$$\int_{\sqrt{2n\pi}}^{\sqrt{(2n+1)\pi}} \sin u^2 \, du < \sqrt{(2n + 1)\pi} - \sqrt{2n\pi} \le \frac{\sqrt{\pi}}{\sqrt{2n + 1} + \sqrt{2n}} ;$$

hence, that the limit of the difference is zero. By the Nested Interval Principal (Section A1-5), y has a limit as t approaches ∞. Similar results hold for x (By the calculus of functions of a complex variable, it can be shown that $\lim_{t\sim\infty} x = \lim_{t\sim\infty} y = \frac{\sqrt{\pi}}{4}$). Thus the functions $f : t \longrightarrow x$, $g : t \longrightarrow y$ have graphs as depicted.



$$f : t \longrightarrow \int_0^t \cos u^2 \, du , \quad g : t \longrightarrow \int_0^t \sin u^2 \, du .$$

The two graphs may then be used together to sketch out the general spiral character of the curve. Observe further, from

$$\frac{d^2y}{dx^2} = [\frac{d}{dt}(\frac{dy}{dx})] \Big/ \frac{dx}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{2t}{[\cos t^2]^3}$$

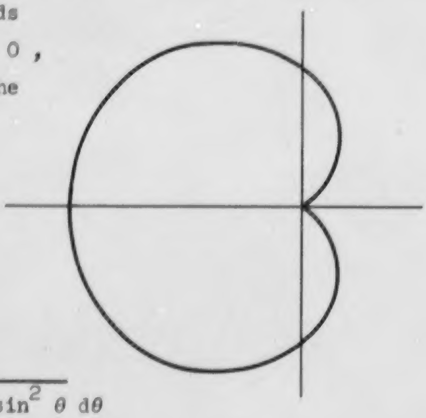
so that the curve is convex between successive extrema of x, flexed upward between a minimum and the ensuing maximum, downward between a maximum and the ensuing minimum.

(d) The cardioid,

$$\begin{cases} x = \cos\theta\,(1 - \cos\theta) \\ y = \sin\theta\,(1 - \cos\theta) \end{cases}, \qquad (0 \le \theta \le 2\pi),$$

or, in polar coordinates, $\rho = 1 - \cos\theta$.

Here $\vec{r}'(t) = (-\sin\theta + \sin 2\theta, \cos\theta - \cos 2\theta)$, hence for $-\pi \le \theta \le \pi$, only $\theta = 0$ yields a null value of r'. At $\theta = 0$, there is a cusp; i.e., since the first component of $\vec{r}'(t)$ is odd and the second component even, the tangent reverses direction. Use the result of Number 5 to obtain



$$s = \int_0^\theta \sqrt{(1 - \cos\theta)^2 + \sin^2\theta}\, d\theta$$

$$= \int_0^\theta \sqrt{2(1 - \cos\theta)}\, d\theta = \int_0^\theta \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta$$

$$= 4(1 - \cos\frac{\theta}{2}), \qquad (0 \le \theta \le 2\pi).$$

Now express θ in terms of s:

$$\theta = 2\arccos(1 - \frac{s}{4}) = \arccos[2(1 - \frac{s}{4})^2 = 1]$$

$$= \arccos[1 - s + \frac{s^2}{8}]$$

$$= \arcsin[\frac{(4 - s)}{2\sqrt{2}}\sqrt{s(1 - \frac{s}{8})}].$$

In terms of s the parametric equations become

$$\begin{cases} x = s(1 - \frac{s}{8})(1 - s + \frac{s^2}{8}) \\ y = \frac{4 - s}{2\sqrt{2}}\,[s(1 - \frac{s}{8})]^{3/2} \end{cases} .$$

(e) $$\begin{cases} x = \dfrac{2t}{1 + t^2} \\ y = \dfrac{1 - t^2}{1 + t^2} \end{cases} .$$

Identify this curve.

Observe that $x^2 + y^2 = 1$. This is the unit circle with the tangent of half the central angle as parameter. The point $(0,-1)$ is not covered except in the limit as t approaches ∞. (Compare Section 10-3, Formula (9)). From $\vec{r}\,'(t) = \dfrac{2}{(1 + t^2)^2}(1 - t^2, -2t)$ (that the curve is a circle, is visible at this point; for $\vec{r}(t) \cdot \vec{r}\,'(t) = 0$, hence, $D[\vec{r}(t)]^2 = 0$ and $[\vec{r}(t)]^2$ is constant), obtain $|\vec{r}\,'(t)| = \dfrac{2}{1 + t^2}$ so that the tangent always exists. For the arclength, obtain

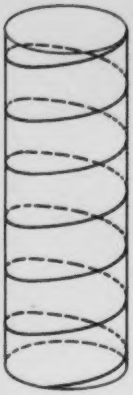
$$s = \int_0^t |\vec{r}\,'(t)|\,dt = 2 \arctan t \ ;$$

whence, $x = \sin s$, $y = \cos s$.

(f) The three-dimensional helix,

$$\begin{cases} x = a \cos t \\ y = a \sin t \\ z = t \end{cases} , \qquad (a > 0) .$$

Here $\vec{r}\,'(t) = (-a \sin t, a \cos t, 1)$ and $|\vec{r}\,'(t)| = c$, where $c = \sqrt{a^2 + 1}$. Thus the tangent always exists and $s = ct$. Consequently,



$$\begin{cases} x = \frac{a}{c} \cos \frac{s}{c} \\ y = \frac{a}{c} \sin \frac{s}{c} \\ z = \frac{s}{c} \end{cases} .$$

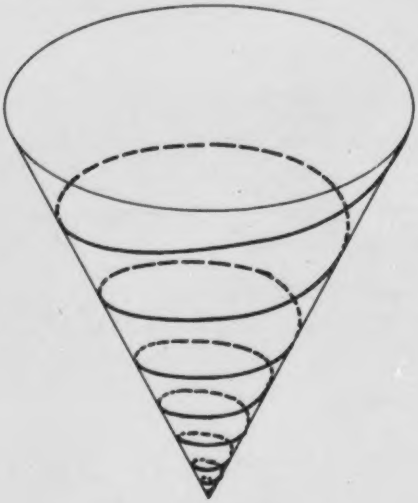
(g) The conical helix,

$$\begin{cases} x = at \cos t \\ y = at \sin t \\ z = t \end{cases} , \qquad (a > 0) .$$

From $\vec{r}'(t) = (a \cos t - at \sin t, a \sin t + at \cos t, 1)$, obtain $|\vec{r}'(t)| = \sqrt{a^2t^2 + (a^2 + 1)}$; whence

$$s = \frac{1}{2} t\sqrt{a^2t^2 + c^2} + \frac{c^2}{a^2} \log (at + \sqrt{a^2t^2 + c^2}) ,$$

where $c = \sqrt{a^2 + 1}$.



7. Show how to define the integral $\int_a^b \vec{f}(t)dt$ by the method of Riemann sums. Prove that this is equivalent to integrating component by component.

Let $\sigma = \{t_0, t_1, \ldots, t_n\}$ be any partition of $[a,b]$ and let τ_k be any number in $[t_{k-1}, t_k]$ for $k = 1, 2, \ldots, n$. Form the sum

$$\vec{S} = \sum_{k=1}^{n} \vec{f}(\tau_k)(t_k - t_{k-1}) .$$

We say that $\vec{I}$ is the integral of $\vec{f}$ over $[a,b]$ and write $\vec{I} = \int_a^b \vec{f}(t)dt$ if for every positive c, there exists a $\delta > 0$ such that $|\vec{S} - \vec{I}| < \epsilon$ whenever $\nu(\sigma) < \delta$, independently of the particular choice of partition or the numbers τ_i.

Now set $\vec{f}(t) = (f(t), g(t), h(t))$ and if the integrals of the components exist, denote them by A_x, A_y, A_z, respectively. First assume that the integral of $\vec{f}$ exists. Then let δ and ϵ be defined as above. It follows that, if $\nu(\sigma) < \delta$, then

$$\left|\sum_{k=1}^{n} f(\tau_k)(t_k - t_{k-1}) - I_x\right| = |S_x - I_x|$$

$$\leq \sqrt{(S_x - I_x)^2 + (S_y - I_y)^2 + (S_z - I_z)^2}$$

$$\leq |\vec{S} - \vec{I}| < \epsilon .$$

Consequently, the integral of f exists and is equal to I_x. The same argument applies to the other components.

Conversely, if the component functions are separately integrable, then given any ϵ it is possible to find a δ_x such that $|S_x - A_x| < \epsilon$ whenever $\nu(\sigma_x) < \delta_x$, where S_x has the form

$$S_x = \sum_{k=1}^{n} f(\tau_{k,x})(t_{k,x} - t_{k-1,x}) .$$

For the corresponding expressions for the other components we have, similarly,

$$|S_y - A_y| < \epsilon \quad \text{whenever} \quad \nu(\sigma_y) < \delta_y$$

and

$$|S_z - A_z| < \epsilon \quad \text{whenever} \quad \nu(\sigma_z) < \delta_z$$

Now, take $\delta = \text{Min}\{\delta_x, \delta_y, \delta_z\}$. Let σ be any partition with $\nu(\sigma) < \delta$, take $\sigma_x = \sigma_y = \sigma_z = \sigma$ and also choose the same intermediate points for the three sums, namely, $\tau_{k,x} = \tau_{k,y} = \tau_{k,z} = \tau_k$. Then,

$$\begin{aligned} |\vec{S} - \vec{A}| &= \sqrt{(S_x - A_x)^2 + (S_y - A_y)^2 + (S_z + A_z)^2} \\ &< \epsilon\sqrt{3} , \end{aligned}$$

where $\epsilon\sqrt{3}$ may be any positive number. Thus $\vec{r}$ is integrable and has the integral $\vec{A} = (A_x, A_y, A_z)$.

Note that integration by components immediately yields the vector counterparts of Theorems 6-4b, c, the Fundamental Theorem and the Substitution Rule.

8. What is the unique continuously differentiable solution of $\vec{r}'(t) = t\vec{A} + \vec{B}$ with the initial condition

$$\vec{r}(0) = \vec{0} \ ?$$

Either employ the last remark in the solution to Number 7 or integrate component by component to obtain

$$\vec{r}(t) = \frac{t^2}{2}\vec{A} + t\vec{B}$$

9. (a) Let the parametric representation of a curve be given in the form $\vec{X} = \vec{r}(s)$ where s is arclength and $\vec{r}$ is three times differentiable. From $|\vec{t}| = \left|\frac{d\vec{X}}{ds}\right| = 1$, it follows from Example 11-5a that $\frac{d\vec{t}}{ds}$ is perpendicular to $\vec{t}$. If $\frac{d\vec{t}}{ds} \neq 0$, the unit vector $\vec{n} = \frac{d\vec{t}}{ds} \Big/ \left|\frac{d\vec{t}}{ds}\right|$ exists. The vector $\vec{n}$ is called the principle normal to the curve. Assuming that $\vec{n}$ exists for all s , prove that the curve is planar if and only if

$$\vec{t} \times \frac{d\vec{n}}{ds} = 0 .$$

If $\vec{t} \times \frac{d\vec{n}}{ds} = 0$, then

$$\frac{d}{ds}(\vec{t} \times \vec{n}) = \frac{d\vec{t}}{ds} \times \vec{n} + \vec{t} \times \frac{d\vec{n}}{ds}$$

$$= \frac{\vec{n} \times \vec{n}}{\left|\frac{dt}{ds}\right|} + \vec{t} \times \frac{d\vec{n}}{ds} = 0 \ .$$

Consequently, for the so-called binormal $\vec{b} = \vec{t} \times \vec{n}$, we have $\frac{d\vec{b}}{ds} = 0$ and $\vec{b}$ is a constant unit vector. Thus

$$\vec{t} \cdot \vec{b} = \frac{d\vec{X}}{ds} \cdot \vec{b} = \frac{d}{ds}(\vec{X} \cdot \vec{b}) = 0 \ ;$$

whence $\vec{X} \cdot \vec{b} = k$, where k is constant; hence, $(\vec{X} - k\vec{b}) \cdot \vec{b} = 0$. It follows from Example 11-4c that, X lies in a plane perpendicular to $\vec{b}$, where $k\vec{b}$ corresponds to the foot of the perpendicular from the origin to the plane and $|k|$ is the length of that perpendicular.

Conversely, if the curve is planar, and $\vec{N}$ is a unit vector perpendicular to the plane, then

$$\vec{X} \cdot \vec{N} = k$$

where k is constant. Consequently,

$$\frac{d\vec{X}}{ds} \cdot \vec{N} = \vec{t} \cdot \vec{N} = 0 \ ,$$

and

$$\frac{d\vec{t}}{ds} = \left|\frac{d\vec{t}}{ds}\right|(\vec{n} \cdot \vec{N}) = 0 \ .$$

Since, $\vec{N}$ is perpendicular to both $\vec{t}$ and $\vec{n}$,

$$\vec{t} \times \vec{n} = \pm \vec{N} \ .$$

Now differentiate with respect to s and use $\vec{n} \times \vec{n} = 0$ to obtain the desired result.

(b) Express this condition in terms of derivatives of $\vec{X}$.

With primes to denote differentiation with respect to s , we have $\vec{t} = \vec{X}'$ and

$$\frac{d\vec{n}}{ds} = \frac{d}{ds}\,\frac{\vec{t}'}{|\vec{t}'|} = \frac{\vec{t}''}{|\vec{t}'|} + \left[\frac{d}{ds}\,\frac{1}{|\vec{t}'|}\right]\vec{t}'$$

$$= \frac{\vec{t}''}{|\vec{t}'|} - \frac{\vec{t}'' \cdot \vec{t}'}{|\vec{t}'|^3}\,\vec{t}' \ ,$$

where the last step uses the result of Number 4(b). Thus, the condition becomes, on multiplication by $|\vec{t}'|$,

$$\vec{X}' \times \left[\vec{X}''' - \frac{(\vec{X}''' \cdot \vec{X}'')\vec{X}''}{|\vec{X}''|^2} \right] = 0$$

where the second vector in the cross-product is the component of $\vec{X}'''$ perpendicular to $\vec{X}''$.

10. We have restricted ourselves to parametrizations of curves $\vec{X} = \vec{r}(t)$, $a \leq t \leq b$, for which the derivative $\vec{r}'$ is continuous with isolated zeros, if any. In general, we say that $\vec{t}$ is the tangent to the curve X_0 , if there is a parametrization $\vec{r}$, with $\vec{X}_0 = \vec{r}(t_0)$, such that

$$\vec{t} = \lim_{t \sim t_0^+} \vec{v}(t) = \lim_{t \sim t_0^-} \vec{v}(t) ,$$

where

$$\vec{v}(t) = \frac{\vec{r}(t) - \vec{r}(t_0)}{t - t_0} \Bigg/ \left| \frac{\vec{r}(t) - \vec{r}(t_0)}{t - t_0} \right| .$$

(a) Show that this definition includes the text case

$$\vec{t} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} .$$

If $\vec{r}'(t_0) \neq 0$ apply the property of the products of limits (5) to obtain

$$\lim_{t \sim t_0^+} \vec{u}(t) = \lim_{t \sim t_0^-} \vec{u}(t)$$

$$= \lim_{t \sim t_0} \left| \frac{\vec{r}(t) - \vec{r}(t_0)}{t - t_0} \right|^{-1} \frac{\vec{r}(t) - \vec{r}(t_0)}{t - t_0}$$

$$= \frac{\vec{r}'(t_0)}{|\vec{r}'(t_0)|} .$$

(b) In the text we have defined parametrizations $\vec{X} = \vec{q}(\tau)$ and $\vec{X} = \vec{r}(t)$ as equivalent if $\vec{r}(\phi(\tau)) = \vec{q}(\tau)$ where ϕ is defined on the domain of $\vec{q}$, has a range in the domain of $\vec{r}$, is increasing, and has a piecewise continuous derivative. Prove that the tangent $\vec{t}$ at X_0 as defined in Part (a) is the same for all equivalent parametrizations.

We shall say that a point t_0 is a regular point of the parametrization $\vec{X} = \vec{r}(t)$ if $\vec{r}'$ is continuous at t_0 and $\vec{r}'(t_0) \neq 0$, otherwise t_0 will be called a singular point. We have restricted ourselves to parametrizations for which the singular points are isolated. Now, set $t = \phi(\tau)$ where t and τ are equivalent. With the possible exception of isolated points we have ϕ' continuous and $\phi'(\tau) \neq 0$. By the Chain Rule

$$\vec{q}'(\tau) = \phi'(\tau)\vec{r}'(t) \neq \vec{0} \tag{1}$$

except perhaps at isolated points. Since $\phi(\tau)$ is increasing, and $\phi'(\tau) \neq 0$ in (1) we have $\phi'(\tau) > 0$; hence, for the tangent,

$$\vec{t} = \frac{\vec{q}'(\tau)}{|\vec{q}'(\tau)|} = \frac{\phi'(\tau)\vec{r}(t)}{|\phi'(\tau)||\vec{r}(t)|} = \frac{\vec{r}(t)}{|\vec{r}(t)|} .$$

Now let τ_0 be one of the isolated points in question. If the curve has a tangent at $\vec{X} = \vec{q}(\tau_0) = \vec{r}(t_0)$ then the right-and-left-sided limits given above exist. Furthermore, since ϕ is increasing, if $\tau \neq \tau_0$, then $\phi(\tau) \neq \phi(\tau_0)$ and

$$\begin{aligned}\frac{\vec{r}(t) - \vec{r}(t_0)}{t - t_0} &= \frac{\vec{q}(\tau) - \vec{q}(\tau_0)}{\phi(\tau) - \phi(\tau_0)} \\ &= \frac{\tau - \tau_0}{\phi(\tau) - \phi(\tau_0)} \; \frac{\vec{q}(\tau) - \vec{q}(\tau_0)}{\tau - \tau_0} \\ &= \frac{1}{\phi'(\bar{\tau})} \; \frac{\vec{q}(\tau) - \vec{q}(\tau_0)}{\tau - \tau_0} ,\end{aligned}$$

where $\bar{\tau}$ is the mean value given by the Law of the Mean. Since τ_0 is an isolated zero or discontinuity point of ϕ', there is a deleted neighborhood of τ_0 which contains no zeros of ϕ'. If τ is restricted to this neighborhood, then $\phi'(\tau) \neq 0$; hence $\phi'(\bar{\tau}) > 0$ because ϕ is increasing. Consequently, for the vector function $\vec{u}$ given by

$$\vec{u}(\tau) = \frac{\vec{q}(\tau) - \vec{q}(\tau_0)}{\tau - \tau_0} \Big/ \left|\frac{\vec{q}(\tau) - \vec{q}(\tau_0)}{\tau - \tau_0}\right|$$

we have

$$\vec{v}(t) = \frac{|\phi'(\bar{\tau})|}{\phi'(\bar{\tau})} \hat{u}(\tau) = \hat{u}(\tau) .$$

Consequently the right- and left-sided limits for $\vec{u}(\tau)$ at τ_0 are equal to those of $\vec{v}(t)$ at t_0, and since the latter two limits are equal, we conclude that $\vec{t}$ is the same for both parametrizations.

(c) If, in the definition of equivalent parameters ϕ is replaced by a decreasing function, we say that t and τ are "contravalent" parameters, just to have a word for it. Show that contravalent parametrizations orient the curve in opposite senses; that is, if $\vec{t}$ is the tangent for the parametrization $\vec{X} = \vec{r}(t)$ then $-\vec{t}$ is the tangent for $\vec{X} = \vec{q}(\tau) = \vec{r}(\phi(\tau))$.

Observe that if t and τ are contravalent parameters, then t and $-\tau$ are equivalent; i.e., for $\sigma = -\tau$ and $p : \sigma \longrightarrow \vec{q}(-\sigma)$, we have

$$\vec{r}(\psi(\sigma)) = \vec{p}(\sigma)$$

where $\psi : \sigma \longrightarrow \phi(-\sigma)$ is an increasing function. Consequently,

$$\frac{\vec{p}(\sigma) - \vec{p}(\sigma_0)}{\sigma - \sigma_0} = -\frac{\vec{q}(\tau) - \vec{q}(\tau_0)}{\tau - \tau_0},$$

from which the conclusion is immediate.

11. A possible vector generalization of Rolle's Theorem is: Let $\vec{r}$ be differentable on $a \leq t \leq b$ and let $\vec{r}(a) = \vec{r}(b) = \vec{0}$, then there is a point t, $a < t < b$, at which $\vec{r}'(t) = \vec{0}$. Prove or disprove.

The statement is false. The proposition fails for any regular closed curve, for example, the circle (Example 11-5e)

$$\begin{cases} x = c \cos \theta \\ y = c \sin \theta \end{cases}$$

for $0 \leq \theta \leq 2\pi$. See Miscellaneous Exercises, Number 10 for a correct generalization.

Λ12. For $\vec{X} = \vec{r}(t)$ where $\vec{r}$ has a continuous derivative on $[a,b]$, prove that the lengths $P(\sigma)$ of inscribed polygons given by (18) have a least upper bound and that this upper bound is the arclength L given by (19).

Note first that the integral L exists since $|r'(t)|$ is continuous. Furthermore L can be approximated within any tolerance by $P(\sigma)$; consequently, if L is an upper bound it must be least. Now, let

$\sigma = \{t_0, t_1, \ldots, t_n\}$ be any partition of $[a,b]$, and let τ be any refinement of σ. Let $\{u_0, u_1, \ldots, u_p\}$ where $u_0 = t_{k-1}$, $u_p = t_k$ be the partition of the interval $[t_{k-1}, t_k]$ by points of τ. Use the inequality (6) of Section 11-4 for the absolute value of a sum, to obtain

$$
\begin{aligned}
|\vec{r}(t_k) - \vec{r}(t_{k-1})| &= |[\vec{r}(u_p) - \vec{r}(u_{p-1})] + [\vec{r}(u_{p-1}) - \vec{r}(u_{p-2})] \\
&\qquad + \ldots + [\vec{r}(u_1) - \vec{r}(u_0)]| \\
&\leq |\vec{r}(u_p) - \vec{r}(u_{p-1})| + |\vec{r}(u_{p-1}) - \vec{r}(u_{p-2})| \\
&\qquad + \ldots + |\vec{r}(u_1) - \vec{r}(u_0)| .
\end{aligned}
$$

Sum over k to obtain $P(\sigma) \leq P(u)$. Thus the arclength $P(\sigma)$ of the inscribed polygon cannot exceed that obtained by any refinement of the partition. Since by making the norm of the partition fine enough we can ensure $|L - P(u)| < \epsilon$, hence, $P(u) < L + \epsilon$, it follows that

$$P(\sigma) < L + \epsilon$$

for all positive ϵ, and we conclude $P(\sigma) \leq L$.

Λ13. Complete the proof that the arclength integral (19) is the limit of the lengths of inscribed polygons (18) by establishing the following lemma.

Let $\vec{R}(t) = (F(t), G(t), H(t))$ be continuous on $[a,b]$. For each partition $\sigma = \{t_0, t_1, \ldots, t_n\}$ of $[a,b]$ and each choice of ξ_k, η_k, ζ_k in $[t_{k-1}, t_k]$, $(k=1,\ldots,n)$, consider

$$P = \sum_{k=1}^{n} \sqrt{F(\xi_k)^2 + G(\eta_k)^2 + H(\zeta_k)^2}\ (t_k - t_{k-1}) .$$

Under the stated condition,

$$\lim_{\nu(\sigma) \sim 0} P = \int_a^b \sqrt{F(t)^2 + G(t)^2 + H(t)^2}\, dt .$$

From the definition of Riemann integral, the result is established when $\xi_k = \eta_k = \zeta_k$. In order to show that it makes no difference if these numbers are used independently, make use of the property proved in Theorem A7-2: If ϕ is continuous on the closed interval $[a,b]$, then for any positive ϵ, there is a δ such that $|\phi(u) - \phi(v)| < \epsilon$ for any u, v in $[a,b]$ satisfying $|u - v| < \delta$. Given ϵ, if δ_1, δ_2, δ_3 are such error controls for F, G, H, respectively, then $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ may be used for all three. Now we restrict ourselves to partitions σ for which $\nu(\sigma) < \delta$. Compare P with

$$Q = \sum_{k=1}^{n} \sqrt{F(\xi_k)^2 + G(\xi_k)^2 + H(\xi_k)^2}\ (t_k - t_{k-1}) \ .$$

Observe from the inequality for the difference of absolute values, (7) of Section 11-4, and from

$$|(a,b,c)| \leq |(a,0,0)| + |(0,b,0)| + |(0,0,c)|$$

that

$$\begin{aligned} &\left|\sqrt{F(\xi_k)^2 + G(\eta_k)^2 + H(\zeta_k)^2} - \sqrt{F(\xi_k)^2 + G(\xi_k)^2 + H(\xi_k)^2}\right| \\ &\leq \sqrt{[F(\xi_k) - F(\xi_k)]^2 + [G(\eta_k) - G(\xi_k)] + [H(\zeta_k) - H(\xi_k)]^2} \\ &\leq |G(\eta_k) - G(\xi_k)| + |H(\zeta_k) - H(\xi_k)| \\ &< 2\epsilon \ . \end{aligned}$$

Consequently,

$$|P - Q| < \sum_{k=1}^{n} 2\epsilon(t_k - t_{k-1}) \leq 2(b - a)\epsilon \ .$$

For a sufficiently fine subdivision, then, the sum P can be brought within any specified tolerance of a Riemann sum. At the same time the Riemann sum can be made to approximate the integral in the same way. It follows that the integral is the limit of the generalized sums P .

TC11-6. Curves in the Plane.

In Subsection (i) we return to issues raised in Chapters 1 and 6 and give a description of area which is sufficiently general to cover most cases of practical interest. In (ii) we introduce the concept of curvature which is not only geometrically interesting but is useful in many applications including the dynamics of a particle (Chapter 12). In (iii) we make use of the idea of curvature to develope the theory of the evolute and involute. These special curves appear again in Chapter 15. In (iv) we utilize the available insights and techniques to derive one of the principle theoretical results about plane curves, that a plane curve is characterized geometrically by its curvature function $s \longrightarrow \kappa$ where s is arclength.

Note that the area integral (3) will usually be improper because of jump discontinuities in ϕ', but this creates no essential difficulty (see Exercises 10-6b, No. 18). For this integral given in the Form (2) we may extend the functions ϕ and ψ to periodic functions with period $\beta - \alpha$; in that case the specific ends of integration are irrelevant so long as their difference is the period (compare the solution of Exercises 11-6, No. 4).

We justify the Formula (3) for area only for curves for which the domain of $\phi : t \mapsto x$ can be subdivided into intervals on which ϕ is strongly monotone or constant. Such a subdivision may not exist even for smooth curves. Nonetheless, the class of curves for which (3) is justified in the text is sufficiently general to motivate the adoption of (3) as the definition of area for the piecewise smooth curves we consider. Since our purpose is the motivation of this definition we are content to give an intuitive geometrical argument for (3). Still, the argument is essentially complete; all that it lacks is a precisely described labeling procedure to insure that nothing has been left out in the step-by-step approach of the text.

In contrast, note that the curvature κ is not invariant under reflection but changes sign. If a coordinate axis is reversed, the positive sense of rotation (from the x-axis to the y-axis) is then reversed and $\vec{n}$ as defined by (9) becomes the right-pointing normal.

In the definition of the osculating circle as the limit of the circle through three points of the curve, if the points are collinear let the approximating "circle" be the straight line through the points as is the usual convention.

In Example 11-6h, the existence of the derivatives of κ is tacitly assumed. Similarly, in Subsection (iv) we assume that $\vec{r}$ has two continuous derivatives.

Observe in Formula (23), since the values of s will include zero if s is measured from a point on the curve, that there is no loss of generality in assuming zero is in the parameter interval. The only effect of using a different end of integration is a change in the constant α.

Solutions Exercises 11-6

1. Verify that the following curves are simple.

(a) The graph of a continuous function.

Take x as the parameter for the graph of $f : x \longrightarrow y$; namely $\vec{X} = (x,f(x))$. For $x \neq x_0$, then $(x_1,f(x_1)) \neq (x_2,f(x_2))$ since the abscissas are distinct.

(b) A circle.

Let the parametric representation of the circle be $\vec{r}(\theta) = (a \cos , a \sin \theta)$, $0 \leq \theta \leq 2\pi$. The circle is closed since $\vec{r}(0) = \vec{r}(2\pi)$. Divide the circle into the semicircles given by $0 \leq \theta < \pi$ and $\pi \leq \theta < 2\pi$. The semicircles do not intersect, since, except for the endpoint $\theta = 0$, points of the first semicircle lie above the x-axis, and, except for the endpoint $\theta = \pi$, those of the second semicircle lie below. Since the endpoints $(1,0)$ and $(-1,0)$ of the respective semicircles are distinct, the semicircles do not intersect each other. Next we consider the upper semicircle. It cannot intersect itself since the abscissa $\cos \theta$ is a decreasing function of θ . The same holds for the lower semicircle with $\cos \theta$ increasing.

(c) A cardioid (Exercises 11-5, No. 6(d)).

As for the circle above, divide the cardioid into the non-overlapping "semicardioids" given by $0 \leq \theta < \pi$ and $\pi \leq \theta < 2\pi$. Use the polar representation $\rho = 1 - \cos \theta$. The result follows since ρ is an increasing function of θ on the upper semicardioid, decreasing on the lower.

Λ(d) The Cornu spiral (Exercises 11-5, No. 6(c)).

Set

$$x = f(t) = \int_0^t \cos u^2 \, du \ , \ y = g(t) = \int_0^t \sin u^2 \, du \ .$$

Observe by the argument of Exercises 11-5, Number 6(c) that $g(0)$, the minimum value of y for $t \geq 0$ and the maximum for $t \leq 0$, separates the ordinates of the two halves of the curve. Hence we may confine our attention to the domain of nonnegative t. At any given point $a > 0$ of the parameter interval let θ be the angle of inclination of the tangent $(f'(a),g'(a))$. Now rotate the coordinate axes through the angle θ so that the tangent becomes horizontal. In the rotated axis system, the coordinates become

$$\begin{cases} \xi = x\cos\theta + y\sin\theta \\ \eta = -x\sin\theta + y\cos\theta , \end{cases}$$

whence

$$\begin{cases} \xi = \phi(t) = \int_0^t \cos(u^2 - \theta)\,du \\ \eta = \psi(t) = \int_0^t \sin(u^2 - \theta)\,du . \end{cases}$$

Now, by the same argument as that of Exercises 11-5, Number 6(c), $\psi(a)$ is a local extremum for η and for $t > a$ is never reached again. Thus $F(b) \neq F(a)$ for any a and b with $b > a \geq 0$. Hence the curve cannot intersect itself for $t \geq 0$.

2. Obtain a parametric representation for the boundary of the standard region under the graph of f oriented in the positive sense indicated in Figure 11-6c.

Take $t = 0$ at $(b,f(b))$, and arclength as the parameter on the straight segments, and $t = b - x$ as the parameter on the graph of f, where k is constant:

$$r(t) = \begin{cases} (b-t,\, f(b-t)), \text{ for } 0 \leq t \leq b-a \\ (a,\, b-a+f(a)-t), \text{ for } b-a \leq t \leq b-a+f(a) \\ (t-f(a)-b+2a,\, 0), \text{ for } b-a+f(a) \leq t \leq 2(b-a)+f(a) \\ (b,\, t-2(b-a)-f(a)), \text{ for } 2(b-a)+f(a) \leq t \leq 2(b-a)+f(a)+f(b). \end{cases}$$

3. Show how the expression for the signed area of a standard region (1), taken in the direction of increasing t, changes if the orientation is negative.

The expression (1) is obtained for positive parametrizations. If τ is a parameter which yields a negative orientation, then t, where $t = -\tau$, yields a positive orientation (compare Exercises 11-5, No. 10(c)). Consequently, for $\tau_0 = -1$, $\tau_1 = -t_0$, the integral

$$I = -\int_{t_0}^{t_1} \psi(t)\phi'(t)dt$$

$$= -\int_{\tau_1}^{\tau_0} \psi(-\tau)\phi'(-\tau)\,\frac{dt}{d\tau}\,d\tau$$

where $\frac{dt}{d\tau} = 1$. Now, in the direction of increasing τ ,

$$I = \int_{\tau_0}^{\tau_1} \hat{\psi}(\tau)\hat{\phi}'(\tau)d\tau \ .$$

where we have used $\hat{\psi}(\tau) = \psi(-\tau)$ and $\hat{\phi}'(\tau) = D_\tau\phi(-\tau) = -\phi'(-\tau)$. Recall that I is the area associated with the positive orientation of the curve; the area A here corresponds to the negative orientation, that is, $A = -I$

$$A = -\int_{\tau_0}^{\tau_1} \hat{\psi}(\tau)\hat{\phi}'(\tau)d\tau \ .$$

Thus, it is shown that the form of (1) is independent of the parametrization.

4. In the derivation of (1) it was supposed that the part of the boundary which is the graph of the given function corresponds to a subinterval of the domain of parametrization. Show for any arc of a simple closed curve how to modify the parametrization so that the arc corresponds to a subinterval of the domain.

Extend ϕ and ψ to periodic functions with period $\beta - \alpha$ and choose for the parameter interval any interval of length $b - a$ for which the range of $\vec{r}$ includes the arc, e.g., if the arc is defined by $\{X : \vec{X} = r(t)\ ,\ t \in [t_0,\beta] \cup [\alpha,t]\}$ where $\alpha \leq t_1 < t_0 < \beta$, set $\tau = t$ for $t \in [t_0,\beta]$, $\tau = t + \alpha + \beta$ for $t \in [\alpha,t_1]$. Then the arc is given by $\vec{X} = \vec{r}(\tau)$, for $t_0 \leq \tau \leq \beta$ and $\vec{X} = \vec{r}(\tau + \alpha - \beta)$, for $\beta \leq \tau \leq t_1 + \beta + \alpha$.

5. Find the area
   (a) under one arch of the cycloid (Exercises 11-5, No. 6(b)).

From (1), with $\phi(t) = a(t - \sin t)$ and $\psi = a(1 - \cos t)$,

$$A = a^2 \int_0^{2\pi} (1 - \cos t)^2 \, dt$$

where the sign corresponds to the negative orientation of the standard region. Consequently

$$\begin{aligned} A &= a^2 \int_0^{2\pi} (1 - 2 \cos t + \cos^2 t) dt \\ &= a^2 \int_0^{2\pi} \left(\frac{3}{2} - 2 \cos t + \frac{\cos 2t}{2}\right) dt \\ &= 3\pi \, a^2 . \end{aligned}$$

(b) of the interior of the cardioid (Exercises 11-5, No. 6(d)).

From (6),

$$A = \frac{1}{2} \int_0^{2\pi} (1 - \cos \theta)^2 \, d\theta ,$$

and from the result of Part (a),

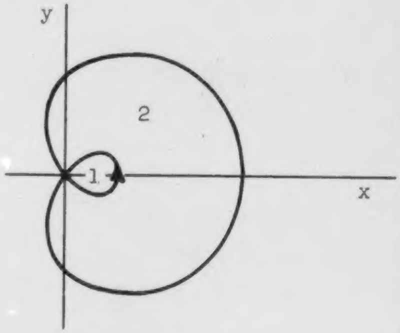
$$A = \frac{3\pi}{2} .$$

6. Sketch the curve given in polar coordinates by $\rho = a \cos \theta - b$, $a > b > 0$, for $0 \leq \theta \leq 2\pi$. Use (5) or an equivalent formula to compute the "area." The result is not the area in the usual sense. Check the derivation of (5) to see what the formula actually gives.

This curve is called the limaçon.
The sketch depicts the case $a = 2b$.
A blind application of (6) yields

$$A = \frac{1}{2} \int_0^{2\pi} \left(b^2 + \frac{a^2}{2} - 2ab \cos \theta + \frac{a^2}{2} \cos 2\theta\right) d\theta$$

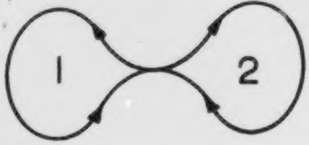
$$= \frac{\pi}{2} (2b^2 + a^2) .$$



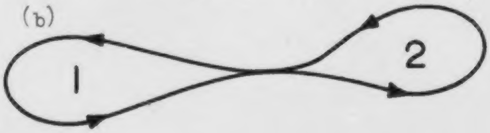
The limaçon is not a simple closed curve but consists of two simple closed curves, both oriented positively. Since the region enclosed by the outer loop consists of the region enclosed by the inner loop (indicated by 1 in the figure) and a region exterior to the inner loop (indicated by 2), the area within the inner loop has been counted twice. Thus $A = 2A_1 + A_2$ where the indices correspond to the regions in the figure.

7. For the following curves what is the "area" as computed by (5)?

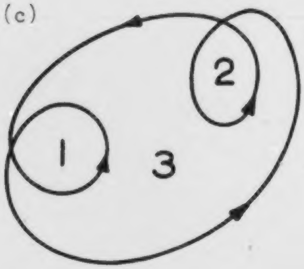
(a)



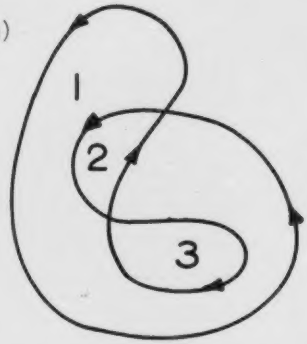
(b)



(c)



(d)



Generalize as far as you can.

Divide up the figure into nonoverlapping subregions as indicated by the labels in each diagram. Let A_i denote the area of subregion i. Count the number of times the subregion is enclosed with sign taken into account.

(a) $A_1 - A_2$

(b) $A_1 + A_2$

(c) $2A_1 + 2A_2 + A_3$

(d) $A_1 + 2A_2$

There are more general situations of interest, but here we show how to treat the case of finitely many selfintersections only. Consider a closed curve given by $\vec{X} = \vec{r}(t)$ for $a \leq t \leq b$ with $\vec{r}(a) = \vec{r}(b)$. A selfintersection, with the possible exception of $t = a$ and $t = b$, occurs when $\vec{r}(t) = \vec{r}(\tau)$ for $\tau \neq t$. By assumption there are only finitely many pairs $\{t, \tau\}$, hence finitely many numbers t for which this happens. Arrange these numbers in increasing sequence,

$$a = t_0 < t_1 < \ldots < t_n = b .$$

For each t_i in this sequence there is at least one t_j, $j \neq i$ such that $\vec{r}(t_j) = \vec{r}(t_i)$. Form all such pairs $\{i, j\}$ with $i < j$, consider

$\min\{j - i\}$ and let $\{i^*,j^*\}$ be a pair for which this minimum is achieved. The closed subcurve $\vec{X} = \vec{r}(t)$, for $t_{i^*} \le t \le t_{j^*}$ is simple, for if a self-intersection $\vec{r}(t_i) = \vec{r}(t_j)$ arose for t_i, $t_j \in [t_{i^*},t_{j^*}]$ and $0 < j - i < j^* - i^*$, then $j^* - i^*$ could not be the minimum. Now take account of the signed area within this subcurve and delete the subcurve from the whole. What remains is still a closed curve but the number of self-intersections has been reduced by at least one. Let the residual curve now be given the continuous parametrization $\vec{X} = q(u)$, where

$$\vec{q}(u) = \begin{cases} \vec{r}(u), \text{ for } a \le u \le t_i^* \\ \vec{r}(u + t_{j^*} - t_{i^*}), \text{ for } t_{i^*} \le u \le b - t_{j^*} + t_{i^*} \end{cases} .$$

The reduction process may now be applied repeatedly to yield a decomposition of the original curve into simple closed curves. The integral is then the sum of the signed area for these.

It is far more complicated to keep track analytically of the number of times the non-overlapping regions are covered.

8. Obtain the expression for the curvature for a curve given in polar coordinates by $\rho = f(\theta)$.

Take θ as the parameter in a cartesian representation:

$$\begin{cases} x = \rho \cos \theta \\ y = \rho \sin \theta . \end{cases}$$

Insert the expressions

$$x' = \rho' \cos \theta - \rho \sin \theta , \; y' = \rho' \sin \theta + \rho \cos \theta$$

and

$$x'' = \rho'' \cos \theta - \theta\rho' \sin \theta - \rho \cos \theta ,$$
$$y'' = \rho'' \sin \theta + 2\rho' \cos \theta - \rho \sin \theta ,$$

in Formula (11) to obtain

$$\kappa = \frac{\rho^2 - \rho \frac{d^2\rho}{d\theta^2} + 2(\frac{d\rho}{d\theta})^2}{[\rho^2 + (\frac{d\rho}{d\theta})^2]^{3/2}} .$$

In the same way, with θ as parameter, we find

$$\kappa = \frac{\frac{d\theta}{d\rho}[1 + (\rho \frac{d\theta}{d\rho})^2] + \rho \frac{d^2\theta}{d\rho^2}}{[1 + (\rho \frac{d\theta}{d\rho})^2]^{3/2}} .$$

9. Find the curvature at each point of the following curves given in cartesian, polar or parametric representation as the notation suggests

(a) $y = x^2$.

$$\kappa = \frac{2}{(1 + 4x^2)^{3/2}} .$$

(b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Use the parametric representation

$$\begin{cases} x = a \cos \theta \\ y = b \sin \theta \end{cases}$$

and apply (11)

$$\kappa = \frac{ab}{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}}$$

$$= \frac{ab}{(a^2 + b^2 - x^2 - y^2)^{3/2}} .$$

(c) $\rho^2 = a^2 \cos 2\theta$.

The curve is a lemniscate. Apply the result of Number 8, as follows. Obtain the first two derivatives with respect to θ of

$$\frac{1}{2} \rho^2 = \frac{1}{2} a^2 \cos 2\theta ;$$

namely

(i) $$\rho\rho' = -a^2 \sin 2\theta$$

and

(ii) $$\rho\rho'' + (\rho')^2 = -2a^2 \cos 2\theta = -2\rho^2 .$$

From (i),

$$(\rho')^2 = \frac{a^4}{\rho^2}\sin^2 2\theta = \frac{a^4}{\rho^2}(1 - \cos^2 2\theta)$$

whence

(iii) $$(\rho')^2 = \frac{a^4}{\rho^2} - \rho^2 .$$

Enter (iii) in (ii) to obtain

(iv) $$\rho\rho'' = -\rho^2 - \frac{a^4}{\rho^2} .$$

Enter (iii) and (iv) in the result of Number 8 to obtain

$$\kappa = \frac{3\rho}{a^2} = \frac{3\sqrt{\cos 2\theta}}{a} .$$

(d) the cycloid (Exercises 11-5, No. 6(b)).

Apply (11) to the given parametric equations.

$$\kappa = \frac{1}{[8a^2(1 - \cos t)]^{1/2}} = - \frac{1}{\sqrt{8ay}} .$$

(e) the Cornu spiral (Exercises 11-5, No. 6(c)).

Apply (11). $\kappa = 2t$.

(f) the cardioid (Exercises 11-5, No. 6(d)).

Apply the result of Number 8.

$$\kappa = \frac{3}{\sqrt{8(1 - \cos \theta)}} = \frac{3}{\sqrt{8\rho}} .$$

(g) $$\begin{cases} x = a\cos^4 \theta \\ y = a\sin^4 \theta \end{cases} .$$

This curve is an arc of a parabola with the line $y = x$ as axis of symmetry. Apply (11) to obtain

$$\kappa = \frac{1}{2a(\cos^4 \theta + \sin^4 \theta)^{3/2}} .$$

Alternatively, eliminate θ to obtain the cartesian form

$$\sqrt{x} + \sqrt{y} = \sqrt{a}$$

and differentiate implicitly with respect to x to obtain

$$y' = -\frac{\sqrt{y}}{\sqrt{x}}$$

and

$$y'' = \frac{\sqrt{x} + \sqrt{y}}{2x^{3/2}} = \frac{\sqrt{a}}{2x^{3/2}} ,$$

whence by (12)

$$\kappa = \frac{\sqrt{a}}{2(x + y)^{3/2}} .$$

The difference in the sign of κ indicates that the two parametrizations are contravalent.

10. Show that the evolute of a cycloid (Exercises 11-5, No. 6(b)) is a cycloid.

Employ the given parametrization to obtain $\vec{r}'(\theta) = a(1 - \cos t , \sin t)$; whence,

$$\vec{t} = \frac{1}{\sqrt{2}}\left(\sqrt{1 - \cos t}\, , \frac{\sin t}{\sqrt{1 - \cos t}}\right) .$$

Now use $1 - \cos t = 2 \sin^2 \frac{t}{2}$ to obtain

$$\vec{t} = (\sin \tfrac{t}{2}, \cos \tfrac{t}{2}) ,$$

a simpler form for computation. Note also that

$$\frac{ds}{dt} = a\sqrt{2(1 - \cos t)} = 2a \sin \frac{t}{2} .$$

Now, since $n = (-\cos \frac{t}{2}, \sin \frac{t}{2})$, we obtain

$$\frac{d\hat{t}}{ds} = \frac{d\hat{t}}{dt} \Big/ \frac{ds}{dt} = -\frac{1}{4a \sin \frac{t}{2}} \hat{n} \; ;$$

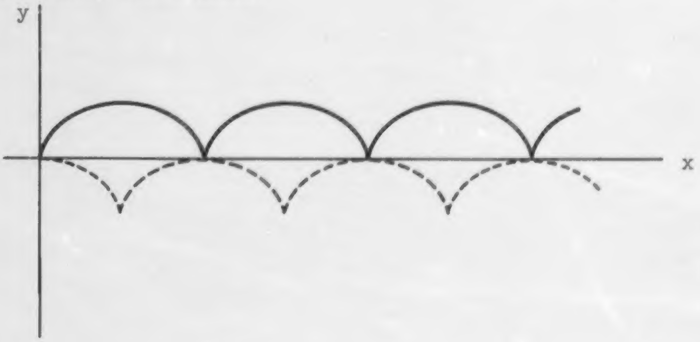
whence, from (10)

$$\kappa = -\frac{1}{4a \sin \frac{t}{2}} \; .$$

From (13) the parametric representation of the evolute is then given by

$$\begin{cases} x = a(t + \sin t) \\ y = -a(1 - \cos t) \; . \end{cases}$$

In the accompanying figure the continuous trace is the cycloid, the dotted trace, its evolute.



To prove the curves are the same, introduce the parameter $\tau = t + \pi$ for the evolute. The parametric representation becomes

$$\begin{cases} x = -a\pi + (\tau - \sin \tau) \\ y = -2a + (1 - \cos \tau) \; . \end{cases}$$

Thus the evolute is a cycloid, translated from the original curve by the vector $(-a\pi, -2a)$.

11. Obtain a cartesian representation for the evolute of an ellipse and sketch the curve.

Use the parametric representation of the ellipse in Number 9(b) to obtain

$$\hat{n} = \frac{-(b \cos \theta \, , \, a \sin \theta)}{\sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}} \; .$$

Take the expression for the curvature from the same source and insert in (13) to obtain the evolute:

$$\begin{cases} x = a \cos \theta - \frac{1}{a}(a^2 \sin^2 \theta + b^2 \cos^2 \theta)\cos \theta \\ y = b \sin \theta - \frac{1}{b}(a^2 \sin^2 \theta + b^2 \cos^2 \theta)\sin \theta \end{cases}$$

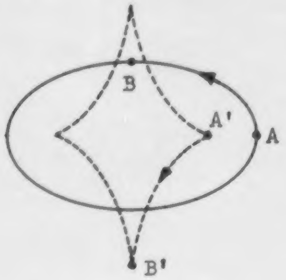
or

$$\begin{cases} ax = (a^2 - b^2)\cos^3 \theta \\ by = -(a^2 - b^2)\sin^3 \theta \ . \end{cases}$$

Solve for $\sin \theta$ and $\cos \theta$, square and add, to obtain

$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3} ,$$

an astroid. In the figure the cusps A' , B' on the evolute (dotted) correspond to the endpoints A , B of the axes of the ellipse where the curvature has extrema.

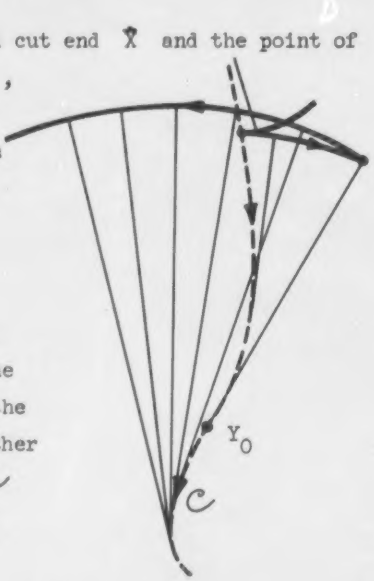


12. The involutes of a curve $\mathcal{C}$ given by $\vec{Y} = \vec{\beta}(\sigma)$ can be drawn by the following simple mechanical construction. Imagine an inextensible thread wrapped tightly around the curve $\mathcal{C}$ on the side opposite the center of curvature. Cut the thread at $\sigma = c$ (compare Equation (17)) and unwrap the thread from the curve while keeping it taut. The condition of tautness requires that the unwrapped thread is pulled out straight and remains tangent to $\mathcal{C}$. Under these conditions show that the two ends of the thread at the cut describe the involute of $\mathcal{C}$.

The length of unwrapped thread between a cut end $\vec{X}$ and the point of tangency $\vec{Y} = \vec{\beta}(\sigma)$ on $\mathcal{C}$ is $\pm(\sigma - c)$, where the sign is positive for the end of the thread unwrapped in the direction of increasing σ and negative for the other end. With this understanding

$$\vec{X} = \vec{Y} + (c - \sigma)\vec{\tau}$$

where $\vec{\tau} = \frac{d\vec{Y}}{ds}$ is the tangent to $\mathcal{C}$ at Y ; but this is just Equation (17a). The figure shows two cusps, one arising at the break in the thread $(\sigma = c)$ and the other corresponding to the point Y_0 where $\mathcal{C}$ has an inflection. The arrows indicate the direction of increasing σ .



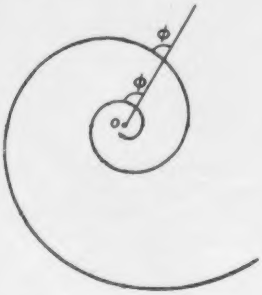
Note that this construction does not give all involutes if the parameter interval is bounded, since the constant c need not be chosen in the domain of $\vec{\rho}$. To obtain an arbitrary involute by this construction it may be necessary to add an extra length of thread. This is what we must do to obtain the ellipse in Number 11 as the involute of the astroid. Note how the construction must be modified when the point of tangency of the thread reaches a cusp of the astroid. Suppose the thread is being unwrapped until it is tangent at a cusp. After the cusp is reached the thread must be wrapped back onto the curve until another cusp is reached. After the latter cusp the thread is unwrapped again; and so we proceed, alternately wrapping and unwrapping until the entire curve is described.

13. The curve $\rho = ce^{a\theta}$, in polar form, has the property that the position vector of a point on the curve makes an angle with the tangent to the curve at the point which is the same for all points.

(a) Verify this property.

In terms of the parameter θ, the position vector is $\vec{X} = ce^{a\theta}(\cos\theta, \sin\theta)$. The tangent vector is

$$\vec{t} = \frac{(a\cos\theta - \sin\theta, a\sin\theta + \cos\theta)}{\sqrt{a^2+1}} .$$



From the formula for the dot product, it follows for the angle ϕ between the two vectors that

$$\cos\phi = \frac{a^2}{\sqrt{a^2+1}} .$$

(b) Show also that the evolute of the equiangular spiral is again the equiangular spiral.

From

$$\vec{X}' = ce^{a\theta}(a\cos\theta - \sin\theta, a\sin\theta + \cos\theta)$$

and

$$\vec{X}'' = ce^{a\theta}(a^2\cos\theta - 2a\sin\theta - \cos\theta, a^2\sin\theta + 2a\cos\theta - \sin\theta)$$

it follows that the radius of curvature is

$$R = ce^{a\theta}\sqrt{a^2 + 1}\ .$$

Consequently $R\hat{n} = (-a \sin\theta - \cos\theta\, , a\cos\theta - \sin\theta)$. Apply (13) to obtain for the evolute

$$\hat{Y} = cae^{a\theta}(-\sin\theta\, , \cos\theta)\ .$$

Set $\theta = \psi + \frac{\pi}{2}$ to obtain

$$Y = ke^{a\psi}\,(\cos\psi\, , \sin\psi)$$

where $k = -cae^{a\pi/2}$.

Now we show that this curve can be a rotation be brought into the original equiangular spiral. If k and c have the same sign, take ψ_0 so that $e^{a\psi_0} = \frac{k}{c}$. Then, in polar form, the curve is given by $\rho = ce^{a(\psi-\psi_0)}$, if k and c have opposite sign then, take ψ_0 so that $e^{a(\psi_0-\pi)} = -\frac{k}{c}$, then

$$Y = ce^{a(\psi+\pi-\psi_0)}(\cos(\psi + \pi)\, , \sin(\psi + \pi))$$

which again yields the polar form

$$\rho = ce^{a(\psi-\psi_0)}\ .$$

14. What is the envelope of the straight line solutions of the differential equation

$$y = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^n\ ?$$

From the result of Example 11-6g, the envelope has the parametric equations

$$\begin{cases} x = -nt^{n-1} \\ y = -(n - 1)t^n \end{cases}$$

whence, for $x < 0$, in any case,

$$y = -(n - 1)\left(-\frac{x}{n}\right)^{\frac{x}{n-1}}\ ;$$

i.e., the curve can be represented by the graph of a constant times a power function. Note that the envelope does satisfy the differential equation.

15. Show that the evolute of the involute of a curve $\mathcal{C}$ is $\mathcal{C}$ itself.

Let $\mathcal{C}$ be given by $\vec{Y} = \rho(\sigma)$. The involute of $\mathcal{C}$ is by (17),

$$\vec{X} = \vec{Y} + (c - \sigma)\,\frac{d\vec{Y}}{d\sigma}\,. \tag{i}$$

The evolute of this curve is given by

$$\vec{Z} = \vec{X} + \frac{1}{\kappa}\,\vec{n} \tag{ii}$$

where $\vec{t}$, $\vec{n}$ and κ are tangent, normal, and curvature for the involute. With the prime indicating differentiation with respect to σ ,

$$\vec{t} = \vec{X}' \Big/ |\vec{X}'| \;,\; \vec{n} = \vec{N} \times \vec{t}$$

where $\vec{N}$ is the unit upward normal to the plane. We have, in the notation of the text,

$$\vec{X}' = (c - \sigma)\,\frac{d\tau}{d\sigma} = (c - \sigma)\gamma\nu$$

and, using $\frac{d\vec{\nu}}{d\sigma} = -\kappa\vec{\tau}$,

$$\vec{X}'' = -(c - \sigma)\gamma^{2}\vec{\tau} + \{\frac{d}{ds}[(c - \sigma)\gamma]\}\vec{\nu}\,.$$

Consequently, from (12) and $\vec{N} = \vec{\tau} + \vec{\nu}$,

$$\kappa = \frac{\vec{N}\cdot(\vec{X}' \times \vec{X}'')}{|\vec{X}'|^{3}} = \frac{(c - \sigma)^{2}\gamma^{3}}{|(c - \sigma)\gamma|^{3}} = \frac{\operatorname{sgn}\gamma}{|c - \sigma|}\,.$$

Now, from $\vec{t} = \frac{(c - \sigma)\gamma}{|(c - \sigma)\gamma|}\,\vec{\nu}$,

$$\vec{n} = \vec{N} \times \vec{t} = \frac{(c - \sigma)\gamma}{|(c - \sigma)\gamma|}\,(N \times \vec{\nu}) = -\,\frac{(c - \sigma)\gamma}{|(c - \sigma)\gamma|}\,\vec{\tau}\;;$$

hence,

$$\frac{\vec{n}}{\kappa} = -(c - \sigma)\vec{\tau} = -(c - \sigma)\frac{d\vec{Y}}{d\sigma}\,.$$

With this, it follows from (i) and (ii) that $\vec{Z} = \vec{Y}$.

16. When does an involute of the evolute of a curve $\mathcal{C}$ coincide with $\mathcal{C}$? If the original curve $\mathcal{C}$ has no cusps then what information does Example 11-6h give about the involutes of the evolute?

Let X_0 be any point of $\mathcal{C}$ and Y_0 the corresponding point of the evolute $\mathcal{E}$. The point X_0 lies on the tangent line $\hat{X}_0 = \hat{Y}_0 + \lambda_0\tau_0$ to the evolute at Y_0. From (17) we see that X_0 lies on the particular involute of $\mathcal{E}$ for which $c = \lambda_0 + \sigma_0$.

If $\mathcal{C}$ has no cusps then from Example 11-6h we see that an involute of $\mathcal{E}$ may have a cusp only if $\sigma - c = 0$; i.e., only if c is in the domain of $\hat{\mathcal{E}} : \sigma \longrightarrow \hat{Y}$.

17. In the text it was asserted that the Solutions (25) of the system of differential equations (20) are all parametric representation of the same geometrical curve. Prove

(a) any member of the family (25) can be obtained from any particular solution by rotation and translation.

Consider first the solution given by $\alpha = x_0 = y_0 = 0$, namely

$$\begin{cases} \xi = \int_0^s \cos \phi \, d\sigma \\ \eta = \int_0^s \sin \phi \, d\sigma . \end{cases}$$

The general solution (25) is given in terms of this special solution by

$$x = x_0 + \int_0^s \cos (\phi + \alpha) d\sigma , \quad y = y_0 + \int_0^s \sin (\phi + \alpha) d\sigma ,$$

or

$$\text{(i)} \qquad \begin{cases} x = x_0 + \xi \cos \alpha - \eta \sin \alpha \\ y = y_0 + \xi \sin \alpha + \eta \cos \alpha . \end{cases}$$

Thus the general solution is obtained from the particular one by rotation through the angle α followed by the translation (x_0,y_0). Since these transformations can be inverted it follows that any member of the family can be obtained from any other by rotations and translations.

(b) given a solution of (20), any transformation of the solution by translation and rotation also is a solution.

Let $\vec{\tau}(s) = (\xi, \eta)$ be a solution of (20). Then $\vec{\rho}(s) = (x,y)$ in (i) represents any curve obtained by rotation and translation. Insert $\vec{\rho}(s)$ in (20) and verify that the equations are satisfied. Thus from (20a),

$$\vec{\rho}\,'(s) = \vec{\tau} = \xi' \cos \alpha - \eta' \sin \alpha \text{ , } \xi' \sin \alpha + \eta' \cos \alpha) \text{ ,}$$

whence $|\vec{\tau}| = \sqrt{(\xi')^2 + (\eta')^2} = |\vec{t}| = 1$. Thus condition (20d) is verified and s is arclength for the transformed curve. Similarly set $\frac{d\tau}{ds} = \gamma \nu$ where γ is curvature for the transformed curve, and observe that

$$\frac{d\vec{\tau}}{ds} = (\xi'' \cos \alpha - \eta'' \sin \alpha \text{ , } \xi'' \sin \alpha + \eta'' \cos \alpha) = \gamma \nu \text{ .}$$

Thus $\gamma \nu$ is merely the vector obtained by rotating $\kappa \vec{n}$ through the angle α . Moreover, since $\vec{\nu}$ is obtained from $\vec{n}$ by the same rotation, $\gamma = \kappa$ (and also (20e) is satisfied). Equation (20c) then follows by (11).

A sophisticated student may observe at once that the statement of the system of equations (20) does not involve the coordinate system; hence, a curve is or is not a solution independently of rotations and translations of the axes. A comparison of the two answers for this exercise should impress the class with the value of the concept of invariance.

18. The catenary (from Latin, catenarius, chain) is the curve assumed by a weighty chain or flexible cable of uniform density when it is hung between two support points. This is the shape of the cable between the towers of a suspension bridge before the deck is laid. The curvature function for the catenary is $k : s \longrightarrow \kappa = \frac{1}{1 + s^2}$. Obtain the equation of a catenary and sketch the curve.

Fix $\alpha = x_0 = y_0 = 0$ in (25). Then from (23),

$$\theta = \int_0^s \frac{1}{1 + \sigma^2}\, d\sigma = \arctan s \text{ .}$$

Consequently, for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$,

$$x = \int_0^s \frac{1}{\sqrt{1+\sigma^2}}\, d\sigma \ , \ y = \int_0^s \frac{\sigma}{\sqrt{1+\sigma^2}}\, d$$

whence,

$$x = \arg\sinh s \ , \ y = \sqrt{1+s^2} \ .$$

Insert $s = \sinh x$ in the expression for y to obtain

$$y = \cosh x \ ,$$

whose graph should be familiar.

19. Let $\vec{X} = \vec{f}(s)$ represent a curve in three-dimensional space. For a space curve it is still true that $\frac{d\vec{t}}{ds}$ is perpendicular to the tangent $\vec{t} = \vec{f}'(s)$, and we define the principal normal $\vec{n}$ as the unit vector in the direction of $\frac{d\vec{t}}{ds}$. The curvature is now defined by $\kappa = \left|\frac{dt}{ds}\right|$ so that Equation (10) is still satisfied.

    (a) Obtain an expression for the curvature of a space curve in terms of any parameter, not necessarily arclength.

    (b) What is the curvature of a helix (Exercises 11-5, No. 6(f)) at any point?

    (c) Investigate whether Equation (11) must hold for a space curve.

Observe that there are infinitely many normal vectors in the plane perpendicualr to $\vec{t}$. In the plane we were able to define a unique normal in terms of $\vec{t}$ alone (by $\vec{n} = \vec{N} \times \vec{t}$ where $\vec{N}$ is the unit upward tangent to the plane); in space we can no longer do so.

(a) Arguing as in Example 11-6f, but with $\kappa = \left|\frac{d\vec{t}}{ds}\right|$, instead of $|\kappa| = \left|\frac{d\vec{t}}{ds}\right|$, obtain instead of (12),

$$\kappa = \frac{|\vec{X}' \times \vec{X}''|}{|\vec{X}'|^3} \ .$$

(b) Use the result of (a). From

$$\vec{X} = (a \cos t \, , a \sin t \, , t)$$

obtain

$$\vec{X}' = (-a \sin t \, , a \cos t \, , 1) \ ,$$

$$|X'| = \sqrt{a^2 + 1} \ ,$$

$$\vec{X}'' = (-a \cos t , -a \sin t , 0) ;$$

whence

$$\kappa = \frac{a^2}{\sqrt{a^2 + 1}} .$$

(c) Equation (11) can be shown to be a necessary and sufficient condition that the curve be planar. To show that (11) need not be satisfied for a space curve, consider the helix. We have

$$\vec{n} = (-\cos t , -\sin t , 0)$$

$$\frac{d\vec{n}}{ds} = \frac{1}{\sqrt{1 + a^2}}(\sin t , -\cos t , 0)$$

which is definitely not collinear with t .

Solutions Miscellaneous Exercises

1. (a) Show that the field of complex numbers is a vector space over the real numbers.

Since the complex numbers form a field (Exercises A1-2, No. 11) they automatically satisfy the addition laws A1, 2, 3, 4 of Section 11-2. Since the real numbers are a subset of that field, the multiplication laws M1, 2 and distributive laws D1, 2 are also satisfied.

(b) Show that the set of positive real numbers, $\mathcal{R}^+$, is a vector space over the field $\mathcal{R}$ of all real numbers where vector addition is defined as ordinary multiplication of real numbers (for p_1 , $p_2 \in \mathcal{R}^+$ the vector sum is p_1p_2) and scalar multiplication is defined as exponentiation (for a scalar $\alpha \in \mathcal{R}$ and a vector $p \in \mathcal{R}^+$, the "product of α with p is p^α).

For any p , q , $r \in \mathcal{R}^+$ and α , $\beta \in \mathcal{R}$ the vector space postulates are satisfied, as follows,

A1: $pq = qp$

A2: $(pq)r = p(qr)$

A3: $p \cdot 1 = p$

A4: $p \cdot \frac{1}{p} = 1$

M1: $p^1 = p$

M2: $(p^\alpha)^\beta = p^{(\alpha\beta)}$

D1: $p^{\alpha+\beta} = p^\alpha \cdot p^\beta$

D2: $(p \cdot q)^\alpha = p^\alpha \cdot q^\alpha$

2. (a) Draw the segments from the right angled vertex of a right triangle to the trisection points of the hypotenuse. Prove that the sum of the squares of the segments is proportional to the square of the hypotenuse and find the constant of proportionality.

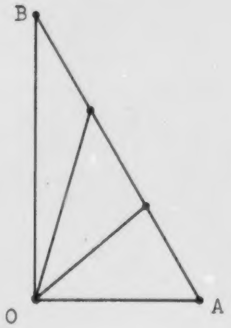
Let O be the right-angled vertex and denote the other two vertices by A and B. The trisection points are given by

$$\frac{\vec{A} + 2\vec{B}}{3} \quad \text{and} \quad \frac{2\vec{A} + \vec{B}}{3} .$$

For the sum of the squares, since $\vec{A} \cdot \vec{B} = 0$, we have

$$\frac{1}{9}(5\vec{A}^2 + 5\vec{B}^2 + 8\vec{A} \cdot \vec{B}) = \frac{5}{9}(A^2 + B^2) .$$

The constant of proportionality is $\frac{5}{9}$.



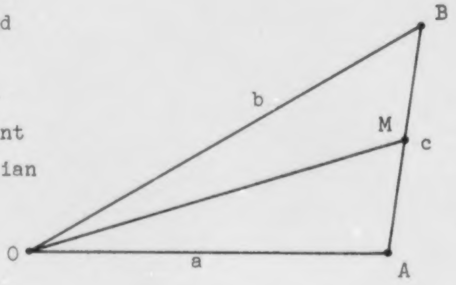
(b) What is the constant of proportionality for the sum of the squares of the segments to the points of section of the hypotenuse into n equal parts?

From the result of Example A3-1g,

$$\frac{1}{n^2}\sum_{k=1}^{n-1} k^2 = \frac{1}{6}\,\frac{(n - 1)(2n - 1)}{n} .$$

3. Given that the sides of a triangle have lengths a, b, c, find the lengths of the medians.

Let O be the vertex of the triangle opposite the side of length c, and let A and B be the remaining vertices with $|A| = a$ and $|B| = b$ (see figure). If M is the midpoint of $\overline{AB}$, then the length of the median $\overline{OM}$ is



$$|\vec{M}| = \left|\frac{\vec{A} + \vec{B}}{2}\right| = \frac{1}{2}\sqrt{(\vec{A} + \vec{B})^2}$$

$$= \frac{1}{2}\sqrt{\vec{A}^2 + \vec{B}^2 + 2\vec{A} \cdot \vec{B}} .$$

From the law of cosines (or $2\vec{A}\cdot\vec{B} = |\vec{A}^2| + |\vec{B}^2| - |\vec{B} - \vec{A}|^2$)

$$2\vec{A}\cdot\vec{B} = a^2 + b^2 - c^2 .$$

Hence,

$$|M| = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2} .$$

The formulas for the other medians are obtained by symmetry.

4. (a) Prove that the cross product is not associative.

It is sufficient to prove nonassociativity for a specific triple of vectors. Let $\{\vec{i},\vec{j},\vec{k}\}$ be a fundamental set of coordinate vectors. Then

$$(\vec{i}\times\vec{j})\times\vec{j} = \vec{k}\times\vec{j} = \vec{i}$$

and

$$\vec{i}\times(\vec{j}\times\vec{j}) = \vec{i}\times\vec{0} = \vec{0} .$$

(b) Under what conditions is the associative law for the cross product of three vectors satisfied?

Suppose

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\times\vec{C} .$$

Expand by Formula (17) of Section 11-4 to obtain

$$(\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C} = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{C}\cdot\vec{B})\vec{A} .$$

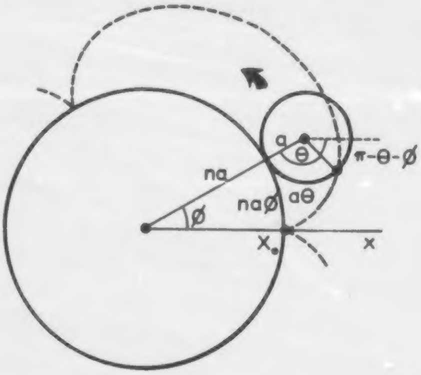
hence,

$$(\vec{A}\cdot\vec{B})\vec{C} = (\vec{C}\cdot\vec{B})\vec{A} .$$

Thus, either $\vec{A}$ and $\vec{C}$ are collinear or $\vec{A}$ and $\vec{C}$ are both perpendicular to $\vec{B}$.

5. The epicycloid of n cusps is the curve traced out by a point of a circle of radius a as it rolls in contact with and outside a fixed circle with radius na (see figure). The hypocycloid of n cusps $(n \geq 3)$ is the curve traced out if the moving circle rolls on the inside of the fixed circle.

   (a) Obtain parametric equations for the epicycloid and the hypocycloid.



Consider the epicycloid first. Let the radius of the rolling circle be a, its center, C, and locate the origin at O. Take the initial position X_0 of the point ϕ at the intersection of the fixed circle with the positive x-axis as indicated in the figure. When the line of centers OC has rotated through an angle ϕ the moving circle has rolled out the arclength $na\phi$. If θ is the angle between $\overline{CX}$ and $\overline{CO}$ then $na\phi = a\theta$, whence $\theta = n\phi$. Take ϕ as parameter. Observe that $\vec{C} = (n + 1)a(\cos\phi, \sin\phi)$ and $\vec{X} = \vec{C} + a(-\cos(\phi + \theta), -\sin(\phi + \theta))$; consequently, the parametric equations of the epicycloid are

$$\begin{cases} x = a[(n + 1)\cos\phi - \cos(n + 1)\phi] \\ y = a[(n + 1)\sin\phi - \sin(n + 1)\phi] . \end{cases}$$

For the hypocycloid (see accompanying figure), observe that

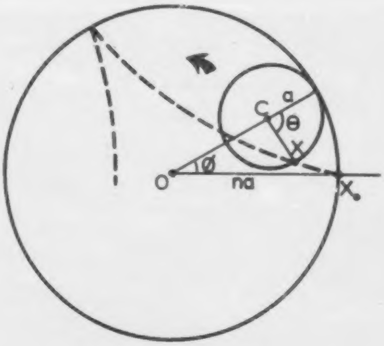
$\vec{C} = (n - 1)a(\cos\phi, \sin\phi)$

and

$\vec{X} = \vec{C} + a(\cos(\theta - \phi), -\sin(\theta - \phi))$

whence,

$$\begin{cases} x = a[(n - 1)\cos\phi + \cos(n - 1)\phi] \\ y = a[(n - 1)\sin\phi - \sin(n - 1)\phi] . \end{cases}$$



   (b) Prove that the epicycloid and hypocycloid of n cusps are simple closed curves.

First we observe for $\vec{X} = \vec{r}(\phi)$ that $\vec{r}(2\pi) = \vec{r}(0)$, thus the curves are closed. Now consider the polar coordinates (ρ,ω) for a point on the epicycloid. From $\omega = \arctan \frac{y}{x} + C$,

$$\frac{d\omega}{d\phi} = \frac{xy' - yx'}{\rho^2} = \frac{(n + 1)(n + 2)a^2(1 - \cos n\phi)}{\rho^2} \geq 0$$

and the zeros of the derivative are isolated. Thus the polar angle ω is an increasing function of ϕ . Furthermore, under the stated conditions $\rho \geq na > 0$ so that the curve cannot intersect itself at the origin, and $\omega = 0$ for $\phi = 0$, $\omega = 2\pi$ for $\phi = 2\pi$ so that no point of the curve except $\vec{r}(0) = \vec{r}(2\pi)$ corresponds to two distinct values of ω .

A similar argument holds for the hypocycloid.

(c) Determine the areas enclosed by the epicycloid and hypocycloid of n cusps.

From the calculation in Part (b) we have for the epicycloid

$$xy' - yx' = (n + 1)(n + 2)a^2(1 - \cos n\phi) .$$

Apply Formula (5) of Section 11-6 with the integral taken over the interval $[0,2\pi]$ to obtain the area

$$A = (n + 1)(n + 2)a^2\pi .$$

Similarly, for the hypocycloid,

$$xy' - yx' = (n - 1)(n - 2)a^2(1 - \cos n\phi) ,$$

and

$$A = (n - 1)(n - 2)a^2\pi .$$

6. Consider a transformation of the plane in which the scales along the coordinates axes are changed independently:

$$(x,y) \longrightarrow (\xi,\eta)$$

where $\xi = ax$, $\eta = by$. Show that if κ is the curvature and θ the angle of inclination of a curve at a point then the transformed curve, at the corresponding point, has the curvature

$$\Upsilon = \frac{ab}{(a^2 \cos^2 \theta + b^2 \sin^2 \theta)^{3/2}} \kappa .$$

If the curve is given by $\hat{X} = (x,y)$ then the transformed curves is $\hat{Y} = (ax,by)$. From $\hat{X}' = (x',y') = |\hat{X}'|(\cos\theta , \sin\theta)$, $\hat{Y}' = (ax',by')$ we have $|\hat{Y}'| = |\hat{X}'|\sqrt{a^2 \cos^2 + b^2 \sin^2 \theta}$. Use this with $\hat{Y}'' = (ax'',by'')$ to obtain the result from Equation (12b) of Section 11-6.

7. Determine the radius of curvature of the evolute of $\mathcal{C}$ in terms of the radius of curvature of $\mathcal{C}$.

In the notation of the text we have $\frac{d\tau}{d\sigma} = \gamma\hat{\nu} = \frac{\hat{\nu}}{Q}$ where Q is the radius of curvature of the evolute. From Section 11-6, (15) and the following equation,

$$\frac{d\tau}{d\sigma} = \frac{d\tau}{ds}\frac{ds}{d\sigma} = -(\operatorname{sgn}\frac{d\kappa}{ds})\ \frac{ds}{d\sigma}\frac{d\hat{n}}{ds}$$

$$= -(\operatorname{sgn}\frac{d\kappa}{ds})(\frac{d}{ds}\frac{1}{\kappa})(\operatorname{sgn}\frac{d}{ds}\frac{1}{\kappa})(-\kappa\hat{t}) \ .$$

Now let R be the radius of curvature of $\mathcal{C}$ and observe that $\hat{\nu} = (\operatorname{sgn}\frac{d\kappa}{ds})\hat{t}$ to obtain

$$\frac{\hat{\nu}}{Q} = (\frac{1}{R}\frac{dR}{ds}\operatorname{sgn}\frac{dR}{ds})\hat{\nu} = \frac{1}{R}\left|\frac{dR}{ds}\right|\hat{\nu}$$

whence

$$Q = R \Big/ \left|\frac{dR}{ds}\right| \ .$$

8. Find the envelope of the family of straight lines given by each criterion:
   (a) The product of the x- and y-intercepts is constant.

The family is given by

(i) $$\frac{x}{\alpha} + \frac{y}{\beta} = 1$$

where $\alpha\beta = k$. In the slope-intercept form of the line

$$y = -\frac{kx}{\alpha^2} + \frac{k}{\alpha} = mx + \sqrt{|km|}\operatorname{sgn}\frac{k}{\alpha}$$

where $m = -\frac{k}{\alpha^2}$. From the discussion of Example 11-6g, the envelope can be given in terms of the parameter α , as follows. Set $f(m) = \sqrt{|km|}\operatorname{sgn}\frac{k}{\alpha}$; then

$$
\begin{aligned}
f'(m) &= \sqrt{\left|\tfrac{k}{m}\right|}\ (\operatorname{sgn} m)\ \operatorname{sgn}\left(\tfrac{k}{\alpha}\right) \\
&= \sqrt{\left|\tfrac{k}{m}\right|}\ \operatorname{sgn} \frac{mk}{\alpha} \\
&= \sqrt{\alpha^2}\ \operatorname{sgn}\left(-\frac{k^2}{\alpha^3}\right) \\
&= -\sqrt{\alpha^2}\ \operatorname{sgn} \alpha \\
&= -\alpha .
\end{aligned}
$$

Hence,

$$
\begin{cases} x = -f'(m) = \alpha \\ y = -mf'(m) + f(m) = 2\dfrac{k}{\alpha} . \end{cases}
$$

Thus, the envelope is the hyperbola

$$xy = 2k .$$

This corresponds to the well-known result that the tangent to a rectangular hyperbola intercepts with the asymptotes a triangle of constant area.

(b) The sum of the x- and y-intercepts is constant c, where $c > 0$.

The family is given by (i) with $\alpha + \beta = c$. In the slope-intercept form,

$$y = \frac{\alpha - c}{\alpha} x + c - \alpha = mx + c \frac{m}{m - 1}$$

where $m = \dfrac{\alpha - c}{\alpha}$. Hence

$$
\begin{cases} x = \dfrac{c}{(m - 1)^2} = \dfrac{\alpha^2}{c} \\ y = \dfrac{mc}{(m - 1)^2} + \dfrac{cm}{m - 1} = \dfrac{\alpha}{c}(\alpha - c) + c - \alpha \\ \quad = \dfrac{(c - \alpha)^2}{c} . \end{cases}
$$

As can be seen by employing a 45° rotation of axes, this curve is a parabola.

9. Obtain a parametric representation of the folium of Descartes given in Exercises 5-7, Number 13. Repeat that exercise in terms of the new representation.

If the parameter t is introduced by $y = xt$ then x^2 can be factored out and x solved for as follows,

$$x^3 + y^3 - 3axy = x^3(1 + t^3) = 3ax^2t = 0 ,$$

where

$$\begin{cases} x = \dfrac{3at}{1 + t^3} \\[2ex] y = \dfrac{3at^2}{1 + t^3} \end{cases} .$$

Note that the origin appears as a point on the curve for $t = 0$ and also in the limit as t appraoches ∞ , also that $t = -1$ represents a gap in the domain of the parameter. The domain of the parameter is made into an interval by using the parameter $u = \dfrac{1 - t}{1 + t}$. Set $t = \dfrac{1 - u}{1 + u}$ above to obtain

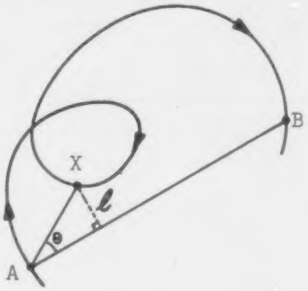
$$\begin{cases} x = \dfrac{(1 - u)(1 + u)^2}{2(3u^2 + 1)} \\[2ex] y = \dfrac{(1 + u)(1 - u)^2}{2(3u^2 + 1)} \end{cases} .$$

The origin is a point of the graph of $u = 1$ where the tangent is horizontal, and $u = -1$, where the tangent is vertical. Elsewhere the slope of the tangent is given by

$$\frac{dy}{dx} = \frac{(1 + u)(1 - 9u + 3u^2 + 3u^3)}{(1 - u)(1 + 9u + 3u^2 - 3u^3)} .$$

10. (a) Prove the following generalization of the Law of the Mean. Let $\mathcal{C}$ be a plane curve given by $\vec{X} = \vec{r}(u)$ on $[a,b]$. If $\vec{r}$ is continuous on the closed interval $[a,b]$, if $\vec{r}\,'$ exists on the open interval (a,b) and is nowhere null, and if $\vec{r}(a) \neq \vec{r}(b)$, then there exists a tangent $\vec{t}_0 = \vec{r}\,'(u_0) \big/ |\vec{r}\,'(u_0)|$ for some u_0 in the open interval which is parallel to the chord joining the endpoints of the curve.

Set $\vec{A} = \vec{r}(a)$, $\vec{B} = \vec{r}(b)$. As for the ordinary Law of the Mean it is convenient to introduce the (signed) distance of the point X from the line AB (see figure). This distance is $\ell = |\vec{X} - \vec{A}| \sin\theta$ where θ is counted positive if $\vec{X}$ is on the left of $\overrightarrow{AB}$ and negative if $\vec{X}$ is on the right. Consequently



$$\ell = \frac{\vec{N}\cdot[(\vec{B} - \vec{A}) \times (\vec{X} - \vec{A})]}{|\vec{B} - \vec{A}|}$$

where $\vec{N}$ is the unit upward normal to the plane. Since $\ell = 0$ for both $t = a$ and $t = b$ it follows by Rolle's Theorem that for some point u_0 of the open interval (a,b) ,

$$\left.\frac{d\ell}{d\sigma}\right|_{u_0} = \frac{\vec{N}\cdot[(\vec{B} - \vec{A}) \times \left|\frac{d\vec{X}}{du}\right|\vec{t}_0]}{|\vec{B} - \vec{A}|} = 0 .$$

Since the cross product must be collinear with $\vec{N}$, it must be null. It follows that $(\vec{B} - \vec{A}) \times \vec{t}_0 = 0$, hence that $\vec{t}_0$ is parallel to the chord $\overline{AB}$ as claimed.

(b) Express the Generalized Law of the Mean in terms of a coordinate representation of $\mathcal{C}$.

Let $\mathcal{C}$ be given by the coordinate representation $x = \phi(u)$, $y = \psi(u)$. There exists a point u_0 in (a,b) such that

$$(\phi'(u_0) , \psi'(u_0)) = \lambda(\phi(b) - \phi(a) , \psi(b) - \psi(a)) .$$

Since at least one component of the vector on the right side of this equation is non-zero, say the first, then λ may be eliminated to give

$$\frac{\psi(b) - \psi(a)}{\phi(b) - \phi(a)} = \frac{\psi'(u_0)}{\phi'(u_0)}$$

for some u_0 in (a,b) . This last statement is the expression of the Generalized Law of the Mean found in most texts.

(c) Prove or disprove the Generalized Law of the Mean for curves in E^3 .

The theorem fails for space curves. Consider the arc of the helix (Exercises 11-5, No. 6f) for $0 \leq t \leq \pi$. We have $\vec{f}(0) = (a,0,0)$, $\vec{f}(\pi) = (-a,0,\pi)$. Thus the chord has the direction of the vector $(-2a,0,\pi)$. At the same time for t_0 in $(0,\pi)$ the tangent vector

$$\vec{X}' = (-a \sin u , a \cos u , 1)$$

has a zero y-component only for $t_0 = \frac{\pi}{2}$ so this is the only place where the tangent can be parallel to the chord. But then $\vec{X}_0{}' = (-a,0,1)$, so parallelism can only occur if $\pi = 2$, which is known to be false.

11. (a) In Exercises 11-6, Number 13 we gave definitions for the principle normal $\vec{n}$ and curvature κ for a space curve $\vec{X} = \vec{f}(s)$. The vector $\frac{d\vec{n}}{ds}$ is perpendicular to $\vec{n}$, but it need not be parallel to $\vec{t}$. We introduce the binormal vector $b = t \times n$. Recall that

(i)
$$\frac{d\vec{t}}{ds} = \kappa\vec{n}$$

and prove that there exists a scalar τ such that

(ii)
$$\frac{d\vec{n}}{ds} = -\kappa\vec{t} + \tau\vec{b}$$

and

(iii)
$$\frac{d\vec{b}}{ds} = -\tau\vec{n} .$$

The scalar τ is called the torsion of the curve. Equations (i), (ii), (iii) which generalize Formulas (10) and (11) of Section 11-6 are the Frenet-Serret equations for the curve.

Since $\frac{d\vec{n}}{ds}$ lies in the plane of $\vec{t}$ and $\vec{b}$ we have

$$\frac{d\vec{n}}{ds} = \sigma\vec{t} + \tau\vec{b}$$

for some scalars σ and τ . Similarly

$$\frac{d\vec{b}}{ds} = \lambda\vec{t} + \mu\vec{n} .$$

Now, differentiate $\vec{t} = \vec{n} \times \vec{b}$ to obtain

$$\begin{aligned} \kappa\vec{n} = \frac{d\vec{t}}{ds} &= \left(\frac{d\vec{n}}{ds} \times \vec{b}\right) + \left(\vec{n} \times \frac{d\vec{b}}{ds}\right) \\ &= \sigma(\vec{t} \times \vec{b}) + \lambda(\vec{n} \times \vec{t}) \\ &= -\sigma\vec{n} - \lambda\vec{b} . \end{aligned}$$

Take the dot product with $\vec{n}$ and $\vec{b}$ to obtain $\sigma = -\kappa$ and $\lambda = 0$. Thus

$$\frac{d\vec{n}}{ds} = -\kappa\vec{t} + \tau\vec{b}$$

which is Equation (ii), and

$$\frac{d\vec{b}}{ds} = \mu\vec{n} .$$

To prove $\mu = -\tau$, differentiate $\vec{b} = \vec{t} \times \vec{n}$ to obtain

$$\begin{aligned} \mu\vec{n} = \frac{d\vec{b}}{ds} &= \left(\frac{d\vec{t}}{ds} \times \vec{n}\right) + \left(\vec{t} \times \frac{d\vec{n}}{ds}\right) \\ &= \kappa(\vec{n} \times \vec{n}) + \vec{t} \times (-\kappa\vec{t} + \tau\vec{b}) \\ &= \tau(\vec{t} \times \vec{b}) = -\tau\vec{n} , \end{aligned}$$

which proves the result.

(b) We have seen that if the curve is plane then $\tau = 0$. Prove, conversely, that if $\tau = 0$, then the curve is plane. (Hint: Show for given functions $\kappa = k(s)$, $\tau = \ell(s)$ that the solutions $\vec{r}$ of the Frenet-Serret equations subject to the initial conditions

(iv) $\qquad \vec{r}(0) = \vec{X}_0 , \ \vec{r}\,'(0) = t_0$

is unique.)

Proceed as in the uniqueness proof in Section 11-6(iv). Let $\vec{r}_1$ and $\vec{r}_2$ be two such solutions, then $\vec{r} = \vec{r}_1 - \vec{r}_2$ satisfies the Frenet-Serret equations together with the homogeneous initial conditions

$$\vec{X}_0 = \vec{t}_0 = \vec{n}_0 = \vec{b}_0 = \vec{0} .$$

where $\vec{X} = \vec{X}_2 - \vec{X}_1$, $\vec{t} = \vec{t}_2 - \vec{t}_1$, etc. Now observe that

$$\begin{aligned} \vec{t} \cdot \frac{d\vec{t}}{ds} &= \kappa(\vec{t} \cdot \vec{n}) \\ \vec{n} \cdot \frac{d\vec{n}}{ds} &= -\kappa(\vec{t} \cdot \vec{n}) + \tau(\vec{b} \cdot \vec{n}) \\ \vec{b} \cdot \frac{d\vec{b}}{ds} &= -\tau(\vec{b} \cdot \vec{n}) , \end{aligned}$$

and add, to obtain

$$2\,\frac{d}{ds}(\vec{t}^{\,2} + \vec{n}^{\,2} + \vec{b}^{\,2}) = 0 .$$

From this and the initial conditions, it follows that

$\vec{t}^2 + \vec{n}^2 + \vec{b}^2 = 0$; whence, $\vec{t} = \vec{n} = \vec{b} = \vec{0}$. Since $\vec{r}'(s) = \vec{t} = 0$ it follows that $\vec{r}(s)$ = constant; hence, by the initial condition, $\vec{r}(s) = 0$, and uniqueness is proved.

We have already seen that there exists a plane curve which satisfies the equations with $\tau = 0$, and the initial conditions $\vec{X} = \vec{X}_0$, $\vec{t} = \vec{t}_0$, $\vec{n} = \vec{n}_0$, at $s = 0$ (the last condition merely fixes the plane containing the curve), for we have characterized such a plane curve by the curvature function alone. It follows from the uniqueness theorem that any solution of the equations with $\tau = 0$ must be plane.

Teacher's Commentary

Chapter 12

MECHANICS

TC 12-1. Introduction.

In this chapter and in Chapter 15 we examine the uses of the calculus in a physical science. The objectives of the two chapters are not the same. In Chapter 15, our purpose is to show the breadth of application of mathematics by showing how it enters at every stage in the development of a science. In this chapter we examine a very limited segment of the science of mechanics, essentially only an introduction to the dynamics of a particle, in order to exhibit some of the more significant uses of the methods we have developed in Sections 10-7, 8, 9, and the vector approach of Chapter 11. The mathematical content of this chapter is primarily the solution of linear differential equations, but with enough indication (the pendulum problem) that non-linear problems are also important.

The historical material in this chapter is concerned with the evolution of ideas rather than names, dates, and anecdotes. Its purpose is to provide an opportunity to learn how to ask and analyze questions about the real world with the help of mathematics. For this purpose, it is hard to better the illustration of the creation of mechanics in the hands of Galileo, Kepler, and Newton.

Insofar as mechanics is concerned with the physical world the student must make judgments about the appropriateness of mathematical descriptions for the interpretation of the physical world. This is not an activity for the superior student alone; the middling student with the background of the first eleven chapters will also find the interplay of mathematical and physical ideas illuminating. Questions will naturally arise which go far beyond the material of the text. Such probings should be encouraged, but it is usually necessary to make an effort to overcome a certain anxiety to achieve the freedom to explore and become excited about questions for which it is not clear whether the student and the teacher have the resources to find answers. Every scientist has felt the same trepidation when he dared to push beyond the limitations of his knowledge.

Since our purpose is not primarily to teach physics it will probably be necessary to put bounds on the classroom discussion of physical questions. Although we wish to avoid a superficial approach to science just as much as we wish to avoid superficiality in mathmatics, the text also cannot develop the ideas of mechanics in the detail necessary for a physics course, and it gives only the briefest account of the foundations of mechanics. You may wish to refer students to Physics, Physical Sciences Study Committee, (Heath, Boston 1960) or a text in use in your school. Without sacrificing our emphasis on the mathematics, we hope that the student will perceive the calculus and mechanics as subjects which were born and grew up together and will maintain a sense of the unity of knowledge.

In contrast to Chapter 9, the problems are not meant to illustrate a narrow range of techniques which stem from a single frequently occurring pattern in a broad spectrum of sciences. Rather the problems come out of the development of one science and the student is placed in the position of a creator of mechanics. He does not know beforehand what analysis will be fruitful. He must be ingenious in drawing upon his knowledge, with all its limitations, and perhaps invent analysis if necessary (although no basically new ideas are really needed here). To a large degree, the science poses the questions, and we have followed lines suggested by the science in selecting problems, subject to the restriction that they be approachable by techniques within the comprehension of the student. In attempting solutions he can expect to fail as much as to succeed. The exploratory activity can be instructive and rewarding even if it does not solve the specific problem posed. There is no reason to be discouraged by the inevitable false trails. Even Newton stumbled. Kepler is said to have written on the title page of one of his lesser works, "Even a blind chicken occasionally finds a kernel of corn" --- not self-depreciation, but pleasure at coming through at last. If Kepler could feel like a blind chicken in search of a grain of truth, no student need be ashamed to lose his way in attempting the solution of a significant problem.

With this understanding of the nature of the problems, the class can be excited simply by sharpening their thoughts concerning a challenging problem in open classroom discussion where conjecture and speculation based on physical intuition can and should play as important a role as the mathematical argument which finally clinches matters. The suggested time of one month for coverage of this chapter is meant to allow for such classroom exploration and for detailed general discussion of several of the more demanding problems. Time is not expected to be adequate to cover all of these in depth. We recognize the impossibility of programming such activity with any precision. It is bound to produce unanticipated reactions from the students -- mostly in the form of half-baked ideas, yet we confidently expect that a few students will produce simpler and more insightful attacks on some of the problems than we have provided in this commentary.

Solutions Exercises 12-1

1. (a) Consider an inertial coordinate system, that is, a system in which Newton's laws hold. Let $\vec{r}(t)$ be the path of a particle in the given system and take new coordinates for which the particle path becomes $\vec{p}(t) = \vec{r}(t) + t\vec{v}$ where $\vec{v}$ is a constant vector. Describe what the change of coordinates means. Show that Newton's laws still hold provided forces are the same in both systems. This result is the Galilean Principle of Relativity.

Observe that the velocity of the particle with respect to the new coordinate frame differs from the velocity in the original coordinate frame by the constant vector $\vec{v}$:

$$\vec{p}\,'(t) = \vec{r}\,'(t) + \vec{v} .$$

It follows that the new coordinate frame is moving with the translatory velocity $-\vec{v}$ with respect to the original frame. Note particularly that the new frame may have any fixed orientation with respect to the old frame; that is, there is no relative rotatory motion of the two frames.

Newton's First Law is immediate, since if the particle is moving with constant velocity in the original frame, $\vec{r}\,'(t) = \vec{v}_0$ for all t , then it is moving with constant velocity $p'(t) = \vec{v}_0 + \vec{v}$ in the new frame. Since the acceleration is the same in both frames, Newton's Second Law (1) is the same in both frames. (Note, however, that if m is not constant, say $m = \mu(t)$, then Newton's Second Law in the Form (2) appears to fail. In order to preserve Newton's Second Law, and conservation of momentum as well, we must consider a closed system for which matter is neither entering nor leaving. Thus to treat the flight of a rocket in the second coordinate frame we must add the term $\mu'(t)\vec{v}$ to the force exerted by the ejected matter. This issue is not relevant for the work of this chapter since we have no need to change coordinate frames.) Newton's Third Law remains valid since it is a statement about forces only, and forces are the same in both coordinate frames.

(b) Let $\vec{r}(t)$ be the particle path in an inertial system as in Part (a). Consider a new system in which the path of the particle is given by $\vec{p}(t) = \vec{r}(t) + \vec{q}(t)$. Show that the laws of motion in the new coordinate system are Newton's laws provided we add the inertial force $m\vec{q}''(t)$ to the total of the forces acting on each particle in the system

In the new system the acceleration of the particle is

$$\vec{p}''(t) = \vec{r}''(t) + \vec{q}''(t) ;$$

thus if $m\vec{r}''(t)$ is the force acting on the particle in the original system, in order to satisfy Newton's Second Law (1) in the new system we must postulate a force $m\vec{p}''(t) = m\vec{r}''(t) + m\vec{q}''(t)$. If a particle is moving with constant velocity in the new system, then $\vec{p}''(t) = 0$ for all t and $m\vec{p}''(t) = 0$ so that the force acting on the particle is zero by Newton's Second Law; hence, the first law also is satisfied. Newton's Third Law is unaffected since the forces of interaction between objects, as indicated by their relative motion, are not affected by any change in the coordinate frame (unless mass is transferred between "objects"; see the solution to Part (a)).

(c) What is the force experienced by an astronaut of mass m if the sole external force exerted upon him is the gravitational attraction mg of the earth and his rocket is accelerating upward with acceleration equal to $6g$?

The force experienced by the astronaut is the force in a frame attached to him, namely the sum of the inertial force and the external gravitational force.

In a frame moving with the rocket with respect to which he is at rest,

$$\zeta''(t) = z''(t) - 6g = 0 ,$$

where z is the upward vertical coordinate fixed with respect to the earth, and ζ , with respect to the rocket. Thus, in addition to gravity, he is experiencing a force equal to $-6mg$ (that is, $6mg$ downward); hence, a total force of $7mg$.

T.C. 12-2. Elementary Mechanical Problems.

In this section we see how mechanics could have served as a potent stimulus to the development of calculus and analysis. Here we have a wealth of significant problems which make extensive use of the mathematical background of the preceding text; the reflective student will perceive many more, some within his powers, most not yet. The search for answers to problems related to the ones treated here still motivates much of the current research in analysis.

In the development of the text we have adhered to invariant vector methods in preference to coordinate techniques. In some places such as the solution to Equation (30) the coordinate technique may seem more straightforward (see the solution to No. 19), but we have kept to the vector approach because the solution enlarges the student's insights while the coordinate solution is relatively mechanical. In any case, the vector representation of the problem suggests suitable choices of coordinate frames. Anyone who was forced to learn mechanics in the old style which preferred three component equations to a single vector equation will appreciate the general gain in brevity and clarity. Coordinates are useful for obtaining some kinds of numerical results, of course, and are not to be avoided when they make for simplicity.

Numerical problems connected with the choice of units and changes from one system of units to another may be of great practical concern but they are not relevant to the calculus and any numerical problems we may give avoid such questions.

The shock absorber of a car is supposed to completely damp out all oscillations of the suspension. Thus it corresponds to the damped case $r^2 > \frac{4k}{m}$ of Equation (16). To test whether a shock absorber is working properly we need only hop on the bumper and see whether it recovers from the displacement monotonically.

Solutions Exercises 12-2

1. Show how to choose a fundamental set $\{\vec{i},\vec{j},\vec{k}\}$ for the derivation of (2) with the additional stipulation that $v_{0x} \geq 0$.

We assume an underlying right-handed frame of reference. Set $\vec{k} = \frac{\vec{g}}{|\vec{g}|}$.

Suppose first that $\vec{v}_0$ and $\vec{k}$ are not collinear. We must choose $\vec{j}$

perpendicular to $\vec{k}$ so that $(\vec{v}_0,\vec{j},\vec{k})$ is a right-handed triple. Consequently, take $\vec{j} = \frac{\vec{k} \times \vec{v}}{|\vec{k} \times \vec{v}|}$ and $\vec{i} = \vec{j} \times \vec{k}$. Compare the solution of Exercises 11-4, No. 8.)

2. Show that Equation (6) yields the Solution (1) of Equation (3).

For this purpose, solve for $\frac{dz}{dt}$ in (6) to obtain the separable equation (Section 10-9)

$$\frac{dz}{dt} = \sqrt{2gz + c_3} \ .$$

Thus, obtain for z ,

$$\frac{1}{g} \sqrt{2gz + c_3} = t + c_4 \ ;$$

whence,

$$z = \frac{g}{2} t^2 + c_5 t + c_6 \ . \tag{i}$$

Set $t = 0$ in (i) to obtain $c_6 = z_0$ and set $t = 0$ in

$$\frac{dz}{dt} = gt + c_5 t$$

to obtain $c_5 = v_{0z}$. Enter these values in (i) to obtain, finally,

$$z = \frac{1}{2} gt^2 + v_{0z}t + z_0 \tag{ii}$$

as in (2), where the origin was chosen so that $z_0 = 0$. We do not have to work through the steps of the argument to obtain the constants in (ii), but we must verify directly that (ii) is a solution of (6) with

$$c_2 = \frac{1}{2} m(v_{0z})^2 - mgz_0 \ .$$

To complete the solution, note that we have already shown in the text that the component of $\vec{v}$ perpendicular to $\vec{g}$ is constant. We have only to take the x-axis in the direction of the perpendicular component and integrate to obtain

$$x = v_{0x}t \quad , \quad y = 0 \ .$$

3. (a) Show that in the limit of small air resistance (k approaches zero) that the Solution (8) of (7) approaches the Solution (1) of (3).

Observe that

$$\lim_{k\sim 0} \vec{X} = \left(\lim_{k\sim 0} \frac{e^{-kt} - 1 + kt}{k^2}\right)\vec{g} + \left(\lim_{k\sim 0} \frac{1 - e^{-kt}}{k}\right)\vec{v}_0 + \vec{X}_0 .$$

From the result of Exercises 8-6, Number 1 we have for $t > 0$.

$$1 - kt + \frac{k^2t^2}{2} - \frac{k^3t^3}{6} \leq e^{-kt} \leq 1 - kt + \frac{k^2t^2}{2} .$$

Apply the Squeeze Theorem to obtain the result

$$\vec{X} = \frac{t^2}{2}\vec{g} + t\,\vec{v}_0 + \vec{X}_0 .$$

(b) Shoot a particle upward; will it return to ground faster if encounters air resistance or no?

In the Solutions (1) and (8), set $X_0 = 0$, take $v_{0z} = -c$ where $c > 0$ is the same in both cases, and determine $t > 0$ when z is again zero. Without air resistance,

$$\frac{g}{2}t^2 - ct = 0$$

yields the positive solution

$$t = \frac{2c}{g} .$$

With air resistance, observe that

$$z = \frac{gt}{k} - \left(c + \frac{g}{k}\right)\left(\frac{1 - e^{-kt}}{k}\right) ,$$

and for $t = \frac{2c}{g}$, in particular

$$\begin{aligned} z &= \frac{c}{k}(1 + e^{-2ck/g}) + \frac{g}{k^2}\left(1 - \frac{2ck}{g}\right) \\ &= g(1 + e^{-2ck/g})\left[\frac{ck}{g} - \frac{1 - e^{-2ck/g}}{1 + e^{2ck/g}}\right] \\ &= g(1 + e^{-2ck/g})\left[\frac{ck}{g} - \frac{e^{ck/g} - e^{-ck/g}}{e^{ck/g} + e^{-ck/g}}\right] . \end{aligned}$$

In this expression, all but the last factor are clearly positive. The last factor has the form $\lambda - \tanh\lambda$ where $\lambda = \frac{ck}{g} > 0$. But $\lambda - \tanh\lambda$ is increasing since it has the positive derivative $\tanh^2\lambda$, (Equation (6) of Section 8-7), and therefore is positive for $\lambda > 0$, since its value at $\lambda = 0$ is zero. Consequently z is positive for $t = \frac{2c}{g}$ and $k > 0$.

(ii) $$z = \frac{c}{k}(1 + e^{-2ck/g}) + \frac{g}{k^2}(1 - e^{-2ck/g}) .$$

Since $e > 2$, for $\frac{2ck}{g} > 1$ we have $e^{-2ck/g} < \frac{1}{2}$ and

$$z > \frac{c}{k} + \frac{g}{2k^2} > 0 ;$$

thus $z > 0$ for $k > \frac{g}{2c}$. Now, from (ii), z can be equal to zero if and only if

$$\begin{aligned} 0 &= \frac{ck}{g} - \frac{1 - e^{-2ck/g}}{1 + e^{-2ck/g}} \\ &= \frac{ck}{g} - \frac{e^{ck/g} - e^{-ck/g}}{e^{ck/g} + e^{-ck/g}} \\ &= \lambda - \tanh\lambda \end{aligned}$$

where $\lambda = \frac{ck}{g} > 0$. But $\lambda - \tanh\lambda$ has the derivative $\tanh^2\lambda$ from Equation (6) of Section 8-7; hence it is increasing and since the z-axis is directed vertically downward this implies that the particle has already returned to ground level when $t = \frac{2c}{g}$. Thus the particle returns to ground faster if it encounters air resistance.

4. For velocities higher than those for which the derivation of (8) is valid, but lower than the speed of sound, it is found experimentally that the retarding force of the atmosphere is proportional to the square of the velocity,

$$\vec{F}_{ret} = -mk|\vec{v}|\vec{v} .$$

(a) Determine the motion of a particle which moves in a vertical line under the influence only of gravity and air friction.

From Newton's Second Law the equations of motion can be written in the form

(i) $$\frac{dz}{dt} = v$$

(ii) $$\frac{dv}{dt} = g - kv^2 \operatorname{sgn} v \,.$$

The two cases, v directed doward $(v > 0)$, and v idrected upward $(v < 0)$, must be treated separately. For algebraic simplicity introduce the constant $\lambda^2 = \frac{k}{g}$. We then have for $v > 0$

(iiia) $$\frac{dv}{dt} = g(1 - \lambda^2 v^2) \,,$$

a separable equation which has the solution (see Section 10-1, Formula (10))

(iiib) $$v = \begin{cases} \frac{1}{\lambda} \tanh \lambda g(t + c_2) \,, & \text{for } 0 < \lambda v < 1 \,, \\ \frac{1}{\lambda} \coth \lambda g(t + c_2) \,, & \text{for } \lambda v > 1 \,. \end{cases}$$

Thus, if the body is falling with a speed less than $\frac{1}{\lambda}$ it picks up speed and approaches the asymptotic speed $\frac{1}{\lambda}$; if it is falling with a speed greater than $\frac{1}{\lambda}$ it slows to the asymptotic speed $\frac{1}{\lambda}$. For the displacement we have on integrating with respect to t,

(iiic) $$z = \begin{cases} c_3 + \frac{1}{\lambda^2 g} \log \cosh \lambda g(t + c_2) \,, & \text{for } 0 < \lambda v < 1 \,, \\ c_3 + \frac{1}{\lambda^2 g} \log \sinh \lambda g(t + c_2) \,, & \text{for } \lambda v > 1 \,. \end{cases}$$

For $v < 0$, we have

(iva) $$\frac{dv}{dt} = g(1 + \lambda^2 v^2)$$

whence,

(ivb) $$v = \frac{1}{\lambda} \tan \lambda g(t + c_4) \,.$$

Thus if the body is ascending with speed $-v$, it loses speed until the speed reaches zero. From that time on, the body falls and the motion is governed by Equation (iiia) with $0 < v < \frac{1}{\lambda}$. For the displacement, we have on integrating (ivb) with respect to t,

(ivc) $$z = c_5 - \frac{1}{\lambda^2 g} \log \cos \lambda g(t + c_4) \,.$$

(b) Re-examine question 3(b) for this form of air resistance.

As in Number 3(b) let c be the initial speed. In Number 3(b) we have found the time of flight without air resistance to be $\frac{2c}{g}$. For the motion with air resistance take $t = 0$ as the instant when the particle reaches maximum height, i.e., when $v = 0$. Then in the first equation of (iiib) we have $c_2 = 0$, and in (ivb), $c_4 = 0$. At the initial instant t_1 of flight we have $z = 0$, $v = -c$. Consequently in (ivc), $c_5 = \frac{1}{\lambda^2 g} \log \cos \lambda g\, t_1$ and

(va) $$z = \frac{1}{\lambda^2 g} \log(\cos \lambda gt)(\cos \lambda gt_1)$$

and in (ivb)

$$-c = \frac{1}{\lambda} \tan \lambda gt_1 \;;$$

whence

(vb) $$t_1 = -\frac{1}{\lambda g} \arctan \lambda c \;.$$

At the final instant t_2 of flight, we have $z = 0$, hence, from the first equation in (iiic), $c_3 = \frac{1}{\lambda^2 g} \log \cosh \lambda g\, t_2$ and

(via) $$z = \frac{1}{\lambda^2 g} \frac{\cosh \lambda gt}{\cosh \lambda gt_2} \;.$$

To determine t_2 , note that the endpoint of the upward leg of the trajectory given by (va) at $t = 0$ must be the beginning point of the downward leg given by (vib) at $t = 0$. Thus

$$\cos \lambda g\, t_2 = \frac{1}{\cos \lambda g\, t_1} = \sqrt{1 + \lambda^2 c^2} \;;$$

hence

(vib) $$\sinh \lambda g\, t_2 = \lambda c \;.$$

From (vb) and (vib) we obtain for the total time of flight

$$t_2 - t_1 = \frac{1}{\lambda g}(\arctan \lambda c + \arg \sinh \lambda c) \;.$$

Now, observe for $\lambda c > 0$ that $\arctan \lambda c < \lambda c$ and $\operatorname{argsinh} \lambda c < \lambda c$, since $x - \arctan x$ and $x - \operatorname{argsinh} x$ are both increasing functions of x . We conclude that

$$t_2 - t_1 < \frac{2c}{g}$$

which is the same as the result of Number 3(b).

5. (a) Solve the equation of motion (10) for a particle moving under the influence of a linear restoring force without restricting the motion to one dimension.

The Equation (10) may be solved component-by-component to yield

(i) $$\vec{X} = \left(\cos \sqrt{\frac{k}{m}}\, t\right)\vec{X}_0 + \left(\sqrt{\frac{m}{k}} \sin \sqrt{\frac{k}{m}}\, t\right)\vec{v}_0 \ .$$

(b) Show in this case that the path of the particle is an ellipse.

This result is strictly valid only if $\vec{X}_0$ and $\vec{v}_0$ are non-collinear. From (i) the trajectory lies in the plane through the origin parallel to the vectors $\vec{X}_0$ and $\vec{v}_0$. For simplicity fix the x-axis so that $\vec{X}_0 = (x_0,0,0)$ and the y-axis so that $\vec{v}_0 = (\xi_0,\eta_0,0)$. Then, from (i)

$$\begin{cases} x = x_0 \cos \omega t + \omega \xi_0 \sin \omega t \\ y = \omega \eta_0 \sin \omega t \end{cases}$$

where $\omega = \sqrt{\frac{k}{m}}$. Insert $\sin \omega t = \frac{y}{\omega \eta_0}$ in the equation for x to obtain

$$\frac{x}{x_0} - \frac{\xi_0}{\eta_0}\frac{y}{x_0} = \cos \omega t \ .$$

Now square the expressions for $\sin \omega t$ and $\cos \omega t$ and sum to obtain

(ii) $$ax^2 + 2bxy + cy^2 = 1$$

where $a = \frac{1}{x_0^2}$, $b = \frac{-\xi_0}{x_0^2 \eta_0}$, $c = \frac{\xi_0^2}{x_0^2 \eta_0^2} + \frac{1}{\omega^2 \eta_0^2}$.

The discriminant is $b^2 - ac = -\frac{1}{\omega^2 x_0^2 \eta_0^2} < 0$; hence Equation (ii) describes an ellipse.

6. Find the Solution (12) of Equation (11) from the first integral of the motion (13).

From (13),

$$\text{(i)} \qquad \frac{dx}{dt} = \sqrt{v_0^2 + \frac{k}{m}(x_0^2 - x^2)}$$

a separable equation. The solution of (i) is

$$\sqrt{\frac{m}{k}} \arcsin \frac{x}{A} = t + \alpha$$

where $A = \sqrt{\frac{mv_0^2}{x} + x_0^2}$ and α is the constant of integration. Consequently,

$$x = A \sin \sqrt{\frac{k}{m}} (t + \alpha) \ ,$$

which we recognize to have the same form as (12) with α chosen so that

$$\sin \alpha = \frac{x_0}{A} \ , \ \cos \alpha = \frac{v_0}{A} \sqrt{\frac{m}{k}} \ .$$

7. Solve Equation (11) when the force kx is a disturbing force $(k < 0)$ rather than a restoring force.

Set $\frac{k}{m} = -c^2$. Equation (11) then becomes

$$\frac{d^2x}{dt^2} - c^2x = 0 \ ,$$

and has the solution

$$x = x_0 \cosh ct + \frac{v_0}{c} \sinh ct \ ,$$

$$= \frac{1}{2}[(x_0 + \frac{v_0}{c})e^{ct} + (x_0 - \frac{v_0}{c})e^{-ct}] \ .$$

Thus the displacement is unbounded for $t > 0$ unless it should happen that $v_0 = -cx_0$; in that case $\lim_{t\to\infty} x = 0$. (Note that the bounded solution is physically unrealistic, since the slightest perturbation in velocity or displacement will make the solution unbounded.)

8. Use the Green's function technique of Section 10-8(ii) to obtain a particular solution of Equation (17).

From the solution to Exercises 10-8(b), Number 6, the Green's function for the operator L in (17) is

$$G(t,\tau) = \frac{1}{b} e^{-a(t-\tau)} \sin b(t - \tau) \ ,$$

where a and b are defined as for (15) by

$$a = -\frac{r}{2} \ , \quad b = \sqrt{\frac{k}{m} - \frac{r^2}{4}} \ .$$

Now apply Formula (28a) of Section 10-8 to obtain the particular solution

$$x_2 = \int_0^t \frac{F}{b} e^{-a(t-\tau)} \sin b(t - \tau) \cos \omega\tau \, d\tau$$

Observe that

$$\sin b(t - \tau)\cos \omega t = \frac{1}{2} \sin[\omega\tau + b(t - \tau)] - \frac{1}{2} \sin[\omega\tau - b(t - \tau)] \ .$$

Now use the result of Example 10-4e, page 558, to obtain,

$$x_2 = \frac{F e^{-a(t-\tau)}}{2b} \left[\frac{a \sin p(\tau) - (\omega - b)\cos p(\tau)}{a^2 + (\omega - b)^2} - \frac{a \sin q(\tau) - (\omega + b) \cos q(\tau)}{a^2 + (\omega + b)^2}\right] \Bigg|_0^t$$

where $p(\tau) = \omega\tau + b(t - \tau)$ and $q(\tau) = \omega\tau - b(t - \tau)$. The contribution from the lower end of integration consists of terms of the form $c_1 e^{-at} \cos bt$ which can be ignored since they satisfy the reduced equation. From the upper end of integration we find the particular solution,

$$x_1 = \frac{F}{2b}\left[\frac{a \sin \omega t - (\omega - b)\cos \omega t}{a^2 + (\omega - b)^2} - \frac{a \sin \omega t - (\omega + b)\cos \omega t}{a^2 + (\omega + b)^2}\right] \ .$$

Now let α be the coefficient of $\sin \omega t$ and β the coefficient of $\cos \omega t$ in this formula. Verify that

$$\sqrt{\alpha^2 + \beta^2} = A \quad \text{and} \quad \frac{\alpha}{\beta} = \tan \phi$$

where A and ϕ are defined by (21). Thus, apart from an added transient solution of the reduced equation, the Green's function technique yields the particular solution (18).

9. (a) Find the general solution of (17) when $r^2 > 4\frac{k}{m}$, the case corresponding to the nonoscillatory damped solution (16) of the reduced equation.

By the same argument used to derive (18) there is a particular solution of the form $A \cos(\omega t - \phi)$, and since the constants A and ϕ are determined precisely as in the text case we see that they are given by (19) as before. Thus the general solution is

$$x = A \cos(\omega t - \phi) + c_1 e^{-\alpha t} + c_2 e^{-\beta t}$$

where α and β are given as for Equation (16). Note that the regime $r^2 > 4\frac{k}{m}$ corresponds to $c > 2$ in Figure 12-3c, and lies within the domain where no resonant frequency occurs.

(b) Find the general solution of (17) when $r^2 = 4\frac{k}{m}$, the so-called critically damped case.

The solution of the reduced equation is

$$e^{-rt/4}(c_1 + c_2 t)$$

and is transient. The particular solution, given by (18) and (19) as before, is the asymptotic solution.

10. (a) Which is nearer to the natural frequency of the undamped system ($r = 0$) governed by (17), the natural frequency or the resonant frequency of the damped system?

We have $\omega_0 = \sqrt{\frac{k}{m}}$ as the circular frequency of the undamped system, $b = \sqrt{\frac{k}{m} - \frac{r^2}{4}}$ as that of the damped system, and $\omega_r = \sqrt{\frac{k}{m} - \frac{r^2}{2}}$ as the resonant frequency. Thus

$$\omega_r < b < \omega_0$$

and

$$b^2 = \frac{1}{2}(\omega_0^2 + \omega_r^2) .$$

(b) The "width" of the tuning curve $A = f(\omega)$ given by (19b) is a useful concept in broadcasting. If a receiver tuned to a station broadcasting at a given frequency has a sharply peaked tuning curve there will be no significant interference from stations broadcasting at nearby frequencies. A convenient measure of the width is

$$\frac{\omega^+ - \omega^-}{\omega r}$$

where ω^- and ω^+ are respectively the frequencies below and above ω_r where the amplitude falls to value $\frac{\alpha_r}{\nu}$, where $\nu > 1$. Express this measure in terms of the constants of the system (17). Obtain an approximate representation for small c.

Observe that the given ratio is equal to

$$\text{(i)} \qquad \frac{\Omega^+ - \Omega^-}{\Omega r}$$

where $\Omega^\pm = \frac{\omega^\pm}{\omega_0}$. The condition on Ω^+ and Ω^- is that α as given by (22) satisfies $\alpha = \frac{1}{\nu}\alpha_r$. In order to simplify the computation use Ω_r as the parameter instead of c. From (24)

$$c^2 = 2 - 2\Omega_r^2 .$$

Insert this in (22) and (23) to obtain

$$\alpha = \frac{1}{\sqrt{1 - 2\Omega_r^2\Omega^2 + \Omega^4}}$$

and

$$\alpha_r = \frac{1}{\sqrt{1 - \Omega_r^4}} .$$

The condition $\alpha = \frac{\alpha_r}{\nu}$ yields the condition (quadratic in Ω^2),

$$\text{(ii)} \qquad \Omega^4 - 2\,\Omega_r^2\Omega^2 + 1 - \nu^2(1 - \Omega_r^4) = 0$$

which has two positive roots

$$\Omega^\pm = \left[\Omega_r^2 \pm \sqrt{(\nu^2 - 1)(1 - \Omega_r^4)}\right]^{1/2} ,$$

provided $\nu^2 < \frac{1}{1 - \Omega_r^4}$. The desired result is obtained by substituting these roots in (i).

For small c , Ω_r is close to 1 , and for fixed ν , the quantity $\sqrt{(\nu^2 - 1)(1 - \Omega_r^4)}$ is small. Use the tangent approximation (Section 5-7)

$$\sqrt{1 + x} \approx 1 + \frac{x}{2}$$

to obtain

$$\Omega^{\pm} = \Omega_r \left[1 \pm \frac{1}{\Omega_r^2} \sqrt{(\nu^2 - 1)(1 - \Omega_r^4)} \right]^{1/2}$$

$$\approx \pm \frac{\sqrt{\nu^2 - 1}\,\sqrt{1 - \Omega_r^4}}{2\Omega_r} .$$

Further, use

$$\sqrt{1 - \Omega_r^4} = c\sqrt{1 - \frac{c^2}{4}} \approx c(1 - \frac{c^2}{8}) \approx c$$

and $\Omega_r = \sqrt{1 - \frac{c^2}{2}} \approx 1 - \frac{c^2}{4} \approx 1$, to obtain

$$\frac{\Omega^+ - \Omega^-}{\Omega_r} \approx c\sqrt{\nu^2 - 1} .$$

Designers of communications equipment refer to $Q = \frac{1}{c}$ as the "quality factor" or simply the "Q" of a tuning circuit. Thus the quality factor is approximately the reciprocal of the width of the tuning curve when $\nu = \sqrt{2}$.

11. Obtain Formula (19b) with the aid of (19a) to complete the work indicated in the text.

Require $0 < \phi < \pi$ so that $\phi > 0$ and the phase does represent a lag rather than an advance. Then $\sin \phi > 0$, and

$$\sin \phi = \frac{r\omega}{\sqrt{(\frac{k}{m} - \omega^2)^2 + r^2\omega^2}} \quad , \quad \cos \phi = \frac{\frac{k}{m} - \omega^2}{\sqrt{(\frac{k}{m} - \omega^2)^2 + r^2\omega^2}} .$$

(Observe that $\cos\phi$ may be negative so that we do not use arctan to invert (19a).) Enter these results in the equation for A preceding (19a) to obtain (19b).

12. What happens when you attempt to get a first integral of (14) by the method of multiplying by $v = \frac{dx}{dt}$? Consider the variation in time of the energy $E = \frac{mv^2}{2} + \frac{kx^2}{2}$? Is energy conserved?

We find

$$m\left(\frac{dx}{dt}\right)\left(\frac{d^2x}{dt^2}\right) + kx\,\frac{dx}{dt} = \frac{d}{dt}\left[\frac{mv^2}{2} + \frac{kx^2}{2}\right] = -mrv^2 .$$

Since $v^2 = \left(\frac{dx}{dt}\right)^2$ is not in the form of a derivative we do not find a constant of the motion. Since $\frac{dE}{dt} \leq 0$, E is decreasing in time (weakly decreasing if v is 0 for all t , but then there is no motion). In a system with friction it is customary to retain the definition of potential energy for the ideal system without friction. The frictional system is said to be dissipative in contrast to the conservative system without friction.

13. Obtain the general solution for (17) when the applied frequency is equal to the resonant frequency for $r = 0$.

In this case Equation (17) becomes

$$L[x] = (D^2 + \omega_0^{\,2})x = F\cos\omega_0 t .$$

Since $F\cos\omega_0 t$ is itself a solution of the reduced equation the method used to obtain a particular solution for $r > 0$ does not work. The Green's function technique does work, but the method of variation of parameters is computationally simpler. The reduced equation has the general solution $a\cos\omega_0 t + b\sin\omega_0 t$. Attempt a particular solution of the form

$$x_1 = \xi\cos\omega_0 t + \eta\sin\omega_0 t .$$

Note that

(i) $L[x_1] = (\xi'' + 2\omega_0\eta')\cos\omega_0 t + (\eta'' - 2\omega_0\xi')\sin\omega_0 t$

$= F\cos\omega_0 t$

and equate coefficients of the sine and cosine terms on the two sides of the equation to obtain

$$\xi'' + 2\omega_0\eta' = F\cos\omega_0 t \ , \quad \eta'' - 2\omega_0\xi' = 0 \ .$$

Eliminate ξ from this pair of equations to obtain

$$\eta''' + 4\omega_0^2\eta' = 2\omega_0 F \ .$$

Integrate to obtain

(ii) $$\eta'' + 4\omega_0^2\eta = 2\omega_0 Ft + c \ .$$

Since any particular solution will do, take $c = 0$. Observe also, by the method of the text, that a particular solution for (ii) of the form $\alpha t + \beta$ can be found. In this case, $\eta = \frac{Ft}{2\omega_0}$; whence $\xi = 0$ above. Thus the general solution of (i) is

$$(a\cos\omega_0 t + b\sin\omega_0 t) + \frac{Ft}{2\omega_0}\sin\omega_0 t \ .$$

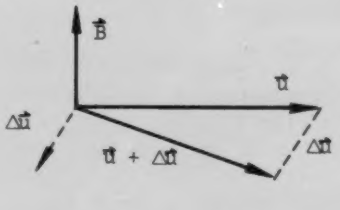
Thus the general solution is the sum of a sinusoidal oscillation and an oscillation which grows without bound.

14. Observe for the undamped spring that the displacement x is an extremum when the velocity $v = 0$ and that the velocity v is an extremum when $x = 0$. Which of these statements is the more surprising?

Since $v = \frac{dx}{dt}$ an extremum when $v = 0$ is usual. On the other hand it is usually false that a zero of x and an extremum of its derivative occur at the same point.

15. In the text it is asserted as "geometrically evident" from (30) that if $q > 0$ the rotation of the direction of $\vec{u}$ is in the negative (clockwise) sense with respect to $\vec{B}$. If it is not evident to you, make it so.

Consider the rotation which occurs in a short time interval Δt . From (30) , $\Delta\vec{u} \approx \frac{q}{m}(\vec{u} \times \vec{B})\Delta t$, thus the sense of rotation is clockwise as seen from the half space into which $\vec{B}$ points, as the figure indicates.



16. For the derivation of (31), let ϕ be any angle measured positively in the counterclockwise sense of rotation from the direction $\vec{u}_0$ to the direction of $\vec{u}$ as seen from the half space into which $\vec{B}$ points. Show that ϕ may be taken as the angle in the definitions of dot product $\vec{u}_0 \cdot \vec{u}$ and cross product $\vec{u}_0 \times \vec{u}$ in (1) and (9) of Section 11-4.

    In the definitions of dot and cross product the angle between $\vec{u}_0$ and $\vec{u}$ was taken without orientation and the measure θ restricted by $0 \leq \theta < \pi$. We may set $\phi = \psi + 2n\pi$ where $0 \leq \psi < 2\pi$. If $0 \leq \psi \leq \pi$ then $\psi = \theta$ and the definitions agree. If $\pi < \psi < 2\pi$ then $\theta = 2\pi - \psi$ and the rotation from the direction from $\vec{u}_0$ to $\vec{u}$ through the angle θ is clockwise as seen from $\vec{B}$. Thus if $\vec{n}$ is the normal in the direction of $\vec{B}$ to the plane of rotation of $\vec{u}$, since $\sin\theta = -\sin\psi = -\sin\phi$

$$\begin{aligned}\vec{u}_0 \times \vec{u} &= -|\vec{u}_0|\,|\vec{u}|\sin\theta\,\vec{n} \\ &= |\vec{u}_0|\,|\vec{u}|\sin\phi\,\vec{n}\ .\end{aligned}$$

    For the dot product, similarly,

$$\begin{aligned}\vec{u}_0 \cdot \vec{u} &= |\vec{u}_0|\,|\vec{u}|\cos\theta \\ &= |\vec{u}_0|\,|\vec{u}|\cos\phi\ .\end{aligned}$$

17. Verify Equation (32) by obtaining the result $k = \dfrac{1}{|\vec{B}|^2}$ given in the preceding text.

    Since $\vec{v}^B$ and $\vec{E}^B$ are perpendicular to $\vec{B}$ for the particular solution (32), the conditions $\vec{v}^B \times \vec{B} = -\vec{E}^B$ and $\vec{v}^B = k(\vec{E}^B \times \vec{B})$, where $k \geq 0$, yield immediately

$$|\vec{v}^B| = \frac{|\vec{E}^B|}{|\vec{B}|} = k|\vec{E}^B|\,|\vec{B}|$$

    from which the result is immediate.

18. In the text we have merely solved (29) for the component of the velocity of the motion perpendicular to $\vec{B}$. Obtain the corresponding component of the displacement vector and give the complete solution of (26).

Add the general solution (31) of the reduced Equation (30) and the particular solution (32) of (29) to obtain

$$\vec{v}^B = (\cos \omega t)\vec{u}_0 - \frac{\sin \omega t}{|\vec{B}|}(\vec{u}_0 \times \vec{B}) + \frac{\vec{E} \times \vec{B}}{B^2}$$

where $\vec{u}_0 = \vec{v}_0^{\,B} - \frac{\vec{E} \times \vec{B}}{B^2}$. Integrate with respect to t to obtain

$$\vec{X}^B = \vec{X}_0^{\,B} + \frac{\sin \omega t}{\omega}\vec{u}_0 + \frac{\cos \omega t}{\omega |\vec{B}|}(\vec{u}_0 \times \vec{B}) + t\,\frac{\vec{E} \times \vec{B}}{B^2} .$$

To obtain the complete description of the motion, add the component (28b) in the direction of $\vec{B}$ to obtain with $\omega = -\frac{q}{m}|B|$,

$$\vec{X} = \vec{X}_0 + t\left(\vec{v}_{0,B} + \frac{\vec{E} \times \vec{B}}{B^2}\right) + \frac{q}{m}t^2 E_B - \frac{\sin \omega t}{\omega}\vec{u}_0 - \frac{\cos \omega t}{\omega}(\vec{u}_0 \times \vec{B}) .$$

19. Solve (29) by introducing an appropriate coordinate frame.

Take a coordinate frame so that

$$\vec{B} = (0,0,B) \quad \text{and} \quad \vec{E}^B = (0,E^B,0)$$

where $B > 0$ and $E^B \geq 0$, and set $\vec{v} = (v_x,v_y,v_z)$. Then (29) yields the system of equations for v_x and v_y

$$\text{(i)} \qquad \begin{cases} mv_x' = qBvy \\ \\ mv_y' = -qBv_x + q\,E^B \end{cases} ,$$

where the prime denotes differentiation with respect to t . Differentiate in the second equation and eliminate v_x by means of the first to obtain

$$v_y'' = -\frac{q^2B^2v_y}{m^2} .$$

The solution for v_y may then be put in the form

$$\text{(ii)} \qquad v_y = c \sin(\alpha t - \phi)$$

where $\alpha = \frac{qB}{m}$. The second equation in (i) yields immediately

$$\text{(iii)} \qquad v_x = c \cos(\alpha t - \phi) - \frac{E^B}{B} .$$

Integrate (ii) and (iii) and use (28b) to obtain the parametric equations of the motion:

$$
\text{(iv)} \quad \left\{ \begin{array}{l}
x = \frac{c_1}{\alpha} \sin(\alpha t - \phi) - \frac{E^B}{B} t + a_x \\[2ex]
y = -\frac{c_1}{\alpha} \cos(\alpha t - \phi) + a_y \\[2ex]
z = \alpha t^2 \frac{\vec{E} \cdot \vec{B}}{B^2} + c_2 t + a_z
\end{array} \right.
$$

where c_1 , c_2 , ϕ , a_1 , a_2 , a_3 are constants of integration.

20. Show that the component of particle motion perpendicular to the magnetic field B is the sum of a uniform straight line motion and a uniform circular motion.

Use either the expression for $\vec{X}$ given in Number 18 or the coordinate representation given in Number 19. From the result of Number 18, for example, write

$$\vec{X}^B = \vec{X}_1 + \vec{X}_2 ,$$

where

$$\vec{X}_1 = \vec{X}_0{}^B + t \frac{\vec{E} \times \vec{B}}{B^2}$$

and

$$\vec{X}_2 = \frac{|u_0|}{\omega}[(\sin \omega t)\vec{i} + (\cos \omega t)\vec{j}]$$

where $\vec{i} = \frac{\vec{u}_0}{|\vec{u}_0|}$ and $\vec{j} = \frac{\vec{u}_0}{|\vec{u}_0|} \times \frac{\vec{B}}{|\vec{B}|}$ are perpendicular unit vectors.

21. Show that the motion of a particle in a constant electromagnetic field where $E = 0$ is a helix (ignore degenerate cases).

Use either the results of Number 18 or Number 19. From Number 19, for example, if $E = 0$, then

$$
\left\{ \begin{array}{l}
x - a_x = c_1 \sin (\alpha t - \phi) \\
y - a_y = c_1 \cos (\alpha t - \phi) \\
z - a_z = kt
\end{array} \right.
$$

Change parameters by substituting $\tau = \frac{\pi}{2} - (\alpha t - \phi)$ and apply the translation $(a_x, a_y, a_z + \frac{1}{\alpha}(\phi + \frac{\pi}{2}))$ to obtain, in essentially the form of Exercises 11-5, Number 6(f),

$$x^* = c_1 \cos \tau \ , \ y^* = c_1 \sin \tau \ , \ z^* = -\frac{\tau}{\alpha} \ .$$

22. Discuss the motion of a particle under the influence of both a constant electromagnetic field and a constant gravitational field.

Newton's Second Law takes the form

$$m \frac{d^2\vec{X}}{dt^2} = m\vec{g} + q(\vec{E} + \frac{d\vec{X}}{dt} \times \vec{B}) \ .$$

This equation can be written in the form of (26) with the introduction of the constant vector $\vec{E}^* = \vec{E} + \frac{m}{q}\vec{g}$; namely

$$m \frac{d^2\vec{X}}{dt} = q(\vec{E}^* + \frac{d\vec{X}}{dt} \times \vec{B})$$

and the solution is given by that of (26) with $\vec{E}$ replaced by $\vec{E}^*$.

23. (a) Solve the equation of rocket motion (35) in one dimension under the assumption that the rate of fuel consumption $-\frac{dM}{dt}$ and exhaust speed $v_e = |\vec{v}_e|$ are constant.

Let the x-axis be oriented in the direction opposite to $\vec{v}_e$ so that $\vec{v} = (v,0,0)$ and $\vec{v}_e = (-v_e,0,0)$. Since $\frac{dM}{dt} = -k$ where the constant k is positive, we have $M = -kt + M_0$. Enter this in (35) to obtain the differential equation

$$(-kt + M_0) \frac{dv}{dt} = kv_e \ ,$$

a separable differential equation which has the solution

$$v = v_0 - v_e \log \frac{M_0 - kt}{M_0} \ .$$

For the displacement x , we obtain with one further integration

$$x = x_0 + (v_0 + v_e)t + v_e(\frac{M_0}{k} - t)\log \frac{M_0 - kt}{M_0} \ .$$

(b) For some purposes it is important that the acceleration not exceed some definite bound, for example, to limit the stress on an astronaut. Suppose the acceleration is set at this bound; replace the assumption in Part (a) by the assumption that the acceleration and v_e are constant and determine the way in which fuel should be consumed to achieve this result.

With a constant acceleration a (35) yields the condition on fuel consumption

$$Ma = -v_e \frac{dM}{dt} .$$

Hence for the expended fuel $m = M_0 - M$, we have

$$(M_0 - m)a = v_e \frac{dm}{dt} ,$$

which yields with the initial condition $m = 0$ at $t = 0$,

$$m = M_0(1 - e^{-at/v_e})$$

as the fuel expended.

24. (a) Solve Equation (36) for the vertical ascent of a rocket in the gravitational field near the surface of the earth ($\vec{g}$ constant).

With the same conventions as the solution of Number 23, $\vec{g} = (-g,C,0)$, and (36) becomes

$$(-kt + M_0)\frac{dv}{dt} = kv_e - (-kt + M_0)g$$

or

(i) $$\frac{dv}{dt} = -g + \frac{kv_e}{M_0 - kt} .$$

Thus the solution for the velocity is

(ii) $$v = v_0 - gt - v_e \log \frac{M_0 - kt}{M_0}$$

and for the displacement

$$x = x_0 + (v_0 + v_e)t - \frac{g}{2} t^2 + v_e(\frac{M_0}{k} - t)\log \frac{M_0 - kt}{M_0} .$$

(b) Consider the motion of Part (a) for a rocket at rest on the ground when $t = 0$. Find the relation between the fuel consumed to the velocity v . Estimate the fuel required to reach a given velocity assuming that it is by far the larger part of the initial mass M_0 .

The initial condition is $v_0 = 0$ in (ii). Use $M_0 - kt = M$ in (ii) to obtain

$$v = -\frac{g}{k}(M_0 - M) - v_e \log \frac{M}{M_0}$$

or, in terms of fuel consumed,

$$v = -\frac{gm}{k} - v_e \log\left(1 - \frac{m}{M_0}\right) , \tag{i}$$

which is the desired relation. Now suppose for a given velocity that $m = M_0(1 - \epsilon)$ where ϵ is small. Then in (i) we have

$$v = \frac{gM_0}{kv_e}(1 - \epsilon) = \log\left(1 - \frac{m}{M_0}\right) ;$$

hence,

$$m = M_0\left[1 - \exp\left(-v - \frac{gM_0}{kv_e}(1 - \epsilon)\right)\right] . \tag{ii}$$

We may ignore ϵ in (ii) as a first approximation (this yields an upper estimate for m). Note from (i) that the rocket will not even lift off the ground at $t = 0$ unless

$$\frac{M_0 g}{kv_e} \leq 1 ,$$

and this condition is assumed in the preceding solution.

(c) Determine the fuel consumption as a function of time under the same assumptions as Part (b) of Number 23.

In the notation of Number 23, the equation of motion becomes

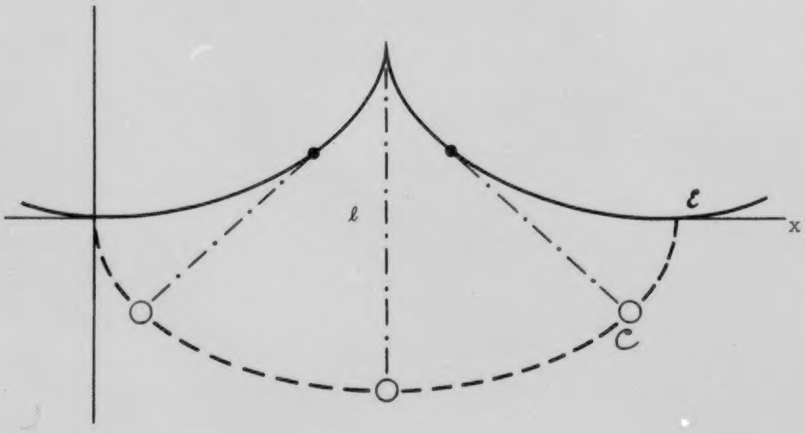
$$M(a + g) = -v_e \frac{dM}{dt} .$$

Thus it is necessary only to replace a by $a + g$ in the solution of Number 23 (b) to obtain

$$m = M_0(1 - e^{-(a + g)t/v_e}) .$$

TC12-3. Constraints. Use of Energy Conservation.

The subject of the cycloidal pendulum deserves more attention than it has received in the text (Example 12-3). Huyghens developed the concepts of evolute and of envelope for a family of straight lines in the courses of his researches on the pendulum. The practical problem in the construction of a cycloidal pendulum is to constrain the particle to a cycloidal path with a minimum of friction. Huyghens produced the ingenious solution shown in the accompanying figure in which a curve $\mathcal{E}$ is used as a guide upon which the pendulum wraps its supporting wire in its upward motion (we used a rod in the text to constrain the motion to two dimensions and also to avoid questions about the straightness of the support wire which require a discussion of the ideal string and the concept of tension). The curve $\mathcal{E}$ is therefore the evolute of the curve $\mathcal{C}$ to which the motion is constrained. For a cycloidal path $\mathcal{E}$ is merely the same cycloid translated (Exercises 11-6, No. 10). For the cycloid the length of the supporting wire is $\ell = 4a$ and by (28), the period is $2\pi\sqrt{\frac{\ell}{g}}$, which is the same as that of the circular pendulum of small amplitude.



The actual design of pendulum clocks has developed along the line of controlling the energy of a circular pendulum, but we cannot help but be impressed by Huyghens's elegant theoretical solution, especially when it is realized that his work preceded the development of the systematic calculus. Huyghens was Leibniz's teacher.

Solutions Exercises 12-3

1. Obtain the Newtonian equations of motion for a particle moving on an inclined plane subject to gravity and frictional forces of the Form (3). Do not assume the z-component of velocity is zero as in the text.

Since the motion is entirely within the plane we may set $\vec{v} = \vec{X}' = (x',0,z')$, where the primes indicate derivatives with respect to t . The Newtonian equations of motion are

$$mx'' = -mg \sin \theta + T_x$$

$$mz'' = T_z \ .$$

From (3) we have

$$T_x = \frac{-(\mu mg \cos \theta)x'}{\sqrt{(x')^2 + (z')^2}} \ , \quad T_z = \frac{-(\mu mg \cos \theta)z'}{\sqrt{(x')^2 + (z')^2}} \ .$$

Thus the equations of motion take the form

$$\text{(i)} \qquad \begin{cases} x'' = -g \sin \theta - \dfrac{(\mu g \cos \theta)x'}{\sqrt{(x')^2 + (z')^2}} \ , \\ z'' = \dfrac{-(\mu g \cos \theta)z'}{\sqrt{(x')^2 + (z')^2}} \ . \end{cases}$$

The student has not been asked to solve these equations, but the equations can be solved for the velocity by techniques already developed in the text and exercises. Set $\vec{X}' = (u,0,w)$. Then (i) can be written in the form

$$\text{(ii)} \qquad \begin{cases} u' = -g \sin \theta - \dfrac{(\mu g \cos \theta)u}{\sqrt{u^2 + w^2}} \ , \\ w' = - \dfrac{(\mu g \cos \theta)w}{\sqrt{u^2 + w^2}} \ . \end{cases}$$

Consequently,

$$\frac{du}{dw} = \frac{u'}{w'} = \frac{\alpha\sqrt{u^2 + w^2} + u}{w} \ ,$$

where $\alpha = \frac{\tan \theta}{\mu}$. Now, take $\lambda = \frac{u}{w}$ and replace u by λw as in Exercises 10-9, Number 3. Then,

$$\frac{du}{dw} = w\,\frac{d\lambda}{dw} + \lambda = \alpha\sqrt{1 + \lambda^2} + \lambda \ .$$

Thus the problem is reduced to the solution of the separable equation

$$\frac{d\lambda}{dw} = \frac{\alpha\sqrt{1 + \lambda^2}}{w} \ ,$$

which has the solution

$$\arg\sinh \lambda = \log c|w|^{\alpha} \ ,$$

where the constant of integration is incorporated in the logarithm. Solve for λ to obtain

$$\lambda = \sinh \log c|w|^{\alpha} \ ,$$

and, since $u = \lambda w$,

(iii) $$u = w \sinh \log c|w|^{\alpha} \ .$$

To complete the solution, enter this expression for u in the second equation of (ii) to obtain

$$\begin{aligned} w' &= -\frac{\mu g \cos\theta \operatorname{sgn} w}{\cosh \log c|w|^{\alpha}} \\ &= -\frac{2\mu g \cos\theta \operatorname{sgn} w}{c|w|^{\alpha} + \dfrac{1}{c|w|^{\alpha}}} \ . \end{aligned}$$

This equation is also separable and has the solution

(iv) $$t = -\frac{1}{2\mu g \cos\theta}\left\{\frac{c|w|^{1+\alpha}}{1 + \alpha} + \frac{|w|^{1-\alpha}}{(1 - \alpha)c}\right\} + k$$

for $\alpha \neq 1$, (we assume $0 \leq \theta < \frac{\pi}{2}$ so that $\alpha \geq 0$) . Observe if $\alpha < 1$, that is $\tan\theta < \mu$, then the z-component of velocity w reaches zero in a finite time and the motion stops. If $\alpha > 1$, then $\lim_{w \to 0} t = \infty$ and the motion asymptotically approaches a direct descent down the plane with the constant acceleration

$$x'' = -g(\sin\theta + \mu\cos\theta) \ .$$

2. Obtain the complete equations of motion for the system consisting of a particle constrained to move on a frictionless wedge which slides on a horizontal plane. Verify that the motion is two-dimensional if the initial velocity is perpendicular to the edge of the wedge.

Use the coordinate frame of the text. Since neither gravity or the force of constraint has a component in the z-direction, Newton's Second Law gives in addition to Equations (5a) and (5b) the condition on the z-component

$$m \frac{d^2z}{dt^2} = 0 \; ;$$

hence, $z = z_0 + v_{0,z}t$. If $v_{0,z} = 0$, the z-component of velocity is zero throughout the motion and the motion is therefore two-dimensional.

3. What is the normal force N exerted by the particle on the frictionless wedge?

From Equations (5a) and (7)

$$N = \frac{M\,mg\cos\theta}{M + m\sin^2\theta} \; ;$$

thus $N = mg$ when $\theta = 0$ and $N = 0$ when $\theta = \frac{\pi}{2}$, which is as it should be.

4. Consider the motion of a particle sliding without friction on a wedge when the wedge slides with the coefficient of friction μ against the horizontal plane.
   (a) Obtain the equations of motion corresponding to (7) and (8) under the assumption that $\frac{d\xi}{dt} > 0$. (Hint: Consider the equation of motion for the y-component of position for the wedge so that the normal force exerted by the plane on the wedge may be taken into account.

Following the hint, indicate the position of the edge of the wedge in the xy-plane by (ξ,η) . To the Conditions (4) and (5) of the text, add the constraint

(i) $$\eta = 0 \; .$$

The components of the forces acting on the wedge in the y-direction are that of gravity, $-Mg$, the push of the particle $-N\cos\theta$, and the supporting force N^* of the horizontal plane. From Newton's Second Law, then,

(ii) $$\frac{d^2\eta}{dt^2} = N^* - Mg - N\cos\theta \; .$$

Insert the constraint (i) in (ii) to obtain

$$N^* = Mg + N\cos\theta\ . \tag{iii}$$

Corresponding to the normal force on the horizontal face of the wedge there is a tangential friction force

$$-\mu N^* \operatorname{sgn} \frac{d\xi}{dt}$$

which must be added to the other horizontal forces on wedge. Under the assumption that $\frac{d\xi}{dt} > 0$ Equation (6) is then replaced by

$$M\frac{d^2\xi}{dt^2} = N\sin\theta - \mu(Mg + N\cos\theta)\ . \tag{iv}$$

Eliminate y and N from Equations (4), (5) and (iv) to obtain the equations

$$\text{(v)}\quad \begin{cases} \dfrac{d^2x}{dt^2} = \dfrac{-Mg\sin\theta\cos\theta\,(1+\mu\tan\theta)}{M + m\sin^2\theta - \mu m\sin\theta\cos\theta} \\[2ex] \dfrac{d^2\xi}{dt^2} = \dfrac{mg\sin\theta\cos\theta\,(1+\mu\tan\theta)(1-\mu\cot\theta)}{M = m\sin^2\theta - \mu m\sin\theta\cos\theta} - \mu g \end{cases}$$

which reduce to (7) and (8) when $\mu = 0$. Although the terms on the right in (v) are complicated in form, they are still constant. Thus the particle still has a constant acceleration to the left and the wedge to the right.

(b) Given that the system is initially at rest, under what conditions will the wedge be set into motion? (Ignore the difference between static and sliding friction).

If the wedge is not set in motion then the forces exerted on the wedge are simply those of a particle sliding on a frictionless inclined plane and a friction force which counterbalances the horizontal thrust of the particle. From Number 2(b) the normal force N_s on the static incline is given by

$$N_s = mg\cos\theta \tag{vi}$$

and the horizontal thrust by $N_s \sin\theta$. From (iii) with $N = N_s$, the vertical force on the base of the wedge is

$$Mg + N_s\cos\theta\ .$$

The wedge will not be set into motion unless the horizontal thrust exceeds the frictional resistance; that is

$$\text{(vii)} \qquad N_s \sin\theta > \mu(Mg + N_s \cos\theta) .$$

(Actually the coefficient in this equation should be μ_s) . Combine (vi) and (vii) to obtain the necessary condition for motion

$$\text{(viii)} \qquad \mu < \frac{m \sin\theta \cos\theta}{M + m\cos^2\theta} .$$

As a supplementary exercise, the student may be asked to show that the Condition (viii) is also sufficient; namely that $\frac{d^2\xi}{dt^2} > 0$ under Condition (viii). This result is not obvious since the normal force, hence also the horizontal thrust, on the incline is not the same for both the static plane and the moving plane as is shown in Part (c). To prove the result, note first that the denominator D in the first term of the expression (v) for $\frac{d^2\xi}{dt^2}$ is positive. From (viii)

$$\mu < \frac{\tan\theta}{1 + \frac{M}{m\cos^2\theta}} < \tan\theta ,$$

consequently

$$\begin{aligned} D &= M + m\sin^2\theta - \mu m \sin\theta\cos\theta \\ &> M + m\sin^2\theta - m\tan\theta\sin\theta\cos\theta \geq M \\ &> 0 . \end{aligned}$$

It follows that the sign of $\frac{d^2\xi}{dt^2} = \frac{N}{D} - \mu g$ has the same sign as $N - \mu g D$, but, by (viii)

$$\begin{aligned} N - \mu g D &= g[m\cos\theta\sin\theta - \mu(M + m\cos^2\theta)] \\ &> 0 . \end{aligned}$$

(c) Determine the order by size of the normal forces exerted by the particle sliding frictionlessly on a wedge for the three cases, stationary wedge, wedge sliding frictionlessly on the horizontal plane, wedge sliding with friction.

Let N_s , N_0 , N_μ , respectively, be the normal forces in the three cases. From (2b) we have

$$N_s = mg \cos \theta$$

and from the solution of Number 3

$$\text{(ix)} \qquad N_0 = \frac{M\, mg \cos \theta}{M + m \sin^2 \theta} = \frac{N_s}{1 + \frac{m}{M} \sin^2 \theta} < N_s .$$

Since the frictional case may plausibly be expected to lie between the other two we anticipate

$$N_0 < N_\mu < N_s$$

and this is the result we prove. From (5a) we have $N_\mu = \frac{-m}{\sin \theta} \frac{d^2x}{dt^2}$. Insert this result in (v) to obtain

$$\text{(x)} \qquad N_\mu = \frac{M\, mg \cos \theta(1 + \mu \tan \theta)}{M + m \sin^2 \theta - \mu m \sin \mu \cos \mu} .$$

If the terms in μ are dropped the numerator in (x) decreases, the denominator increases. Consequently, by (ix), $N_\mu > N_0$.

To obtain the inequality $N_\mu < N_s$ observe that, with D defined as in the solution of Part (b),

$$\begin{aligned} N_s - N_\mu &= \frac{mg \cos \theta}{D}[D - M(1 + \mu \tan \theta)] \\ &= mg \cos \theta \frac{K}{D} , \end{aligned}$$

hence the sign of $N_s - N_\mu$ is the sign of K . Explicity, K is given by

$$\begin{aligned} K &= m \sin^2 \theta - \mu m \sin \theta \cos \theta - \mu M \tan \theta \\ &= m \sin^2 \theta - \mu \tan \theta(M + m \cos^2 \theta) . \end{aligned}$$

Consequently, from (viii), $K > 0$, as we sought to prove.

5. Obtain the energy conservation principle for the system consisting of the particle sliding on a frictionless wedge, Equations (5) - (6). (Hint: Take as the kinetic energy of the system the sum of the kinetic energies for particle and wedge.)

The total kinetic energy is

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{M}{2}\dot{\xi}^2$$

where the dots indicate differentiation with respect to t. From (5a,b) and (6)

$$\begin{aligned}\dot{T} &= m(\dot{x}\,\ddot{x} + \dot{y}\,\ddot{y}) + M\dot{\xi}\,\ddot{\xi}\\ &= -N \sin\theta\,\dot{x} + (N\cos\theta - mg)\dot{y} + N\sin\theta\,\dot{\xi}\ .\end{aligned}$$

Eliminate $\dot{\xi}$ with the aid of (4) by $\dot{\xi} = \dot{x} - \dot{y}\cot\theta$ to obtain

$$\dot{T} = -mg\dot{y}\ ;$$

whence obtain the energy conservation equation

$$T + mgy = k\ .$$

Thus the potential mgy is simply the gravitational potential of the unconstrained particle.

6. What is the magnitude λ of the force of constraint for the pendulum?

The problem is to determine the component of force on the particle normal to the path. Note that the unit tangent to the circle in the direction of increasing θ is $\vec{t} = (\cos\theta, \sin\theta)$, and the normal directed toward the center is $\vec{n} = (-\sin\theta, \cos\theta)$. The position vector of the particle is $\vec{X} = \ell(\sin\theta, -\cos\theta)$; whence, on differentiating twice with respect to t, and applying Newton's Second Law,

$$\begin{aligned}\vec{F} = m\ddot{\vec{X}} &= (m\ell\ddot{\theta}\cos\theta, m\ell\ddot{\theta}\sin\theta) + (-m\ell\dot{\theta}^2\sin\theta, m\ell\dot{\theta}^2\cos\theta)\\ &= m\ell\ddot{\theta}\,\vec{t} + m\ell\dot{\theta}^2\,\vec{n}\ .\end{aligned}$$

Consequently the normal force is directed toward the center; i.e., the rod pulls the bob toward the pivot. For the magnitude of the force we have with $\omega = \dot{\theta}$ and $v = |\dot{\vec{X}}| = \ell\dot{\theta}$

$$\lambda = m\ell\omega^2 = \frac{mv^2}{\ell} = \frac{2mg}{\ell}(z_0 - z)$$

where at the last step we have used the energy principle to give v in terms of z and the elevation z_0 of the stationary points.

7. The text states that it is not immediately obvious how to use the constraint to eliminate the constraint force λ from the equations of motion (15a,b). Show how to do it.

Replace x and z by the parameter θ ; i.e., set $x = \ell \sin \theta$, $y = -\ell \cos \theta$ in the equations of motion, to obtain (as in No. 6)

$$m\ell(\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta) = -\lambda \sin \theta$$
$$m\ell(\ddot{\theta} \sin \theta + \dot{\theta}^2 \cos \theta) - mg = \lambda \cos \theta .$$

Multiply the first of these equations by $\cos \theta$, the second by $\sin \theta$, and add to get

$$m\ell\ddot{\theta} - mg \sin \theta = 0$$

which is essentially the same as (21).

8. Describe the motion of the pendulum when $\theta = \pi$ is a stationary point, that is, when $a = 1$ in (18).

Since $1 + \cos \psi = 2 \cos^2 \frac{\psi}{2}$ we obtain in (18)

$$t = \frac{1}{2} \sqrt{\frac{\ell}{g}} \int_0^\theta \frac{d\psi}{\cos \frac{\psi}{2}} .$$

By the solution of Exercises 10-3, Number 11(b),

$$t = \sqrt{\frac{\ell}{g}} \log \tan \frac{\theta + \pi}{4} \qquad (0 \leq \theta < \pi) ,$$

where $\theta = 0$, $\omega = \sqrt{\frac{2g}{\ell}}$ at $t = 0$. Invert this relation to obtain

$$\theta = -\pi + 4 \arctan e^{t\sqrt{g/\ell}} .$$

From this relation we see that it takes infinite time to reach $\theta = \pi$ from $\theta = 0$ under the given conditions.

9. Estimate the difference between the error of approximation to the amplitude of the exact Solution (20a) of the pendulum problem by that of the approximate Solution (24a).

Choose the initial condition $\theta = 0$, $\omega = \omega_0 > 0$ at $t = 0$ where we require $\omega_0^2 < \frac{2g}{\ell}$ in order to get the periodic Solution (20a). From Equation (16) with $\omega = 0$ we obtain the amplitude α given by

$\cos\alpha = 1 - \frac{\ell}{2g}\omega_0^2$ of which the estimates are to be given. The exact solution for the first quarter cycle is given by

$$t = \sqrt{\frac{\ell}{2g}} \int_0^{\theta} \frac{d\psi}{\sqrt{\cos\psi - \cos\alpha}}$$

and the approximate solution by

$$\hat{\theta} = \omega_0 \sqrt{\frac{\ell}{g}} \sin t \sqrt{\frac{g}{\ell}} .$$

Thus $\cos\alpha = 1 - \frac{\hat{\alpha}^2}{2}$ where $\hat{\alpha}$ is the amplitude of the approximate solution. From Example 7-5b we have from the estimates for the cosine

$$1 - \frac{\alpha^2}{2} \le 1 - \frac{\hat{\alpha}^2}{2} \le 1 - \frac{\alpha^2}{2} + \frac{\alpha^4}{24} .$$

From these inequalities we obtain

$$\alpha\sqrt{1 - \frac{\alpha^2}{12}} \le \hat{\alpha} \le \alpha .$$

Next we use the inequality $1 - x < \sqrt{1 - x}$ for $0 < x < 1$ to obtain with $x = \frac{\alpha^2}{12} < \frac{\pi^2}{12} < 1$,

$$0 \le \alpha - \hat{\alpha} < \frac{\alpha^3}{12} .$$

10. (a) Show that the period of the pendulum as given by (20) is an increasing function of θ_0 .

Let $f : \theta_0 \longrightarrow \tau$ be given by

$$\text{(i)} \qquad \tau = f(\theta_0) = 2\sqrt{\frac{2\ell}{g}} \int_0^{\theta_0} \frac{d\psi}{\sqrt{\cos\psi - \cos\theta_0}} .$$

Consider $f(\lambda\theta_0)$ where $\lambda \ge 1$. With the change of variables $\psi \longrightarrow \lambda\psi$ obtain

$$f(\lambda\theta_0) = 2\sqrt{\frac{2\ell}{g}} \int_0^{\theta_0} g(\psi,\lambda)d\psi$$

where $g(\psi,\lambda) = \dfrac{\lambda}{\sqrt{\cos \lambda\psi - \cos \lambda\, \theta_0}}$. We have $\tau = \int_0^{\theta_0} g(\psi,1)d\psi$.

Consequently we have only to prove that $g(\psi,\lambda) > g(\psi,1)$ for $\lambda > 1$ and apply the result of Exercises 6-4, Number 18(b) which extends a strong inequality between continuous functions to their integrals. Obtain the derivative of g with respect to λ ,

$$D_\lambda\, g(\psi,\lambda) = \frac{\cos \lambda\psi - \cos \lambda\theta_0 + \frac{1}{2}\lambda\psi \sin \lambda\psi - \frac{1}{2}\lambda\theta_0 \sin \lambda\theta_0}{(\cos \lambda\psi - \cos \lambda\theta_0)^{3/2}} .$$

Since $D_\lambda g$ is continuous in λ we need only establish that the derivative is positive when $\lambda = 1$ to show that it is positive in a neighborhood of $\lambda = 1$. Thus we have reduced the problem to showing that the numerator is positive when $\lambda = 1$, that is

$$h(\psi) = \cos \psi + \frac{\psi}{2} \sin \psi > \cos \theta_0 + \frac{\theta_0}{2} \sin \theta_0$$

when $0 < \psi < \theta_0$. For the proof we need only establish that h is a decreasing function. To this end, observe that

$$\begin{aligned} h'(\psi) &= \frac{\psi}{2} \cos \psi - \frac{1}{2} \sin \psi \\ &= \frac{1}{2} \cos \psi(\psi - \tan \psi) < 0 , \end{aligned}$$

where for $0 < \psi < \frac{\pi}{2}$ the result follows from the inequality $\psi < \tan \psi$ (see Exercises 5-3, No. 14(b), also the geometrical argument in Section 4-4), and for $\frac{\pi}{2} \leq \psi < \pi$ the result is immediate from the negative sign of both terms in the first expression for $h'(\psi)$.

In summary, we have proved that the period τ is an increasing function of θ_0 in some neighborhood of θ_0 (i.e., for λ in some neighborhood of 1). But this result is independent of the choice of θ_0 . Hence by the solution of Exercises A4-1, Number 11, τ is an increasing function of θ_0 on the entire interval $(0,\pi)$.

11. Show for a particle oscillating in a potential well that the motion is periodic. Show further that the time to traverse the well from one side to the other is equal to the time to come back.

12-3

The problem is to show under the conditions of the text the integral

$$\sqrt{\frac{m}{2}} \int_a^b \frac{d\sigma}{\sqrt{U(a) - U(\sigma)}}$$

converges. If the integral does converge then so does the integral

$$-\sqrt{\frac{m}{2}} \int_b^a \frac{d\sigma}{\sqrt{U(b) - u(\sigma)}}$$

which corresponds to the time of the back swing, and since $U(a) = U(b) = k$ the two integrals are equal. Since $U(\sigma) < k$ for $a < \sigma < b$, the discontinuities of the integrand $\frac{1}{\sqrt{k - U(\sigma)}}$ occur only at the endpoints $[a,b]$ of the interval. In the neighborhoods of these points apply Theorem 10-6a with a test function of the form $\frac{A}{|\sigma - \sigma_0|^{1/2}}$. For example, at $\sigma_0 = a$, we must have $U'(a) < 0$ since $U'(a) \neq 0$ and $U(\sigma) < U(a)$ for $\sigma > a$. From the definition of derivative,

$$U(\sigma) = U(a) + (\sigma - a)[U'(a) + \epsilon]$$

where $\lim\limits_{\sigma \to 0} \epsilon = 0$. Take a neighborhood of a in which $|\epsilon| < -\frac{1}{2} U'(a)$. Then

$$k - U(\sigma) = U(a) - U(\sigma) \geq \frac{1}{2}|U'(a)|(\sigma - a) ;$$

hence,

$$\frac{1}{\sqrt{k - U(\sigma)}} \leq \frac{2}{|U'(a)|(\sigma - a)} .$$

Similarly in some neighborhood of b,

$$\frac{1}{\sqrt{k - U(\sigma)}} \leq \frac{2}{|U'(b)|(b - \sigma)} .$$

From Theorem 10-6b and Theorem 10-6a it follows that the integral converges.

12. Let $s = 0$ correspond to a maximum of the potential U, with $U(0) = k$, $U'(0) = 0$, and $U''(0) < 0$, where U'' is continuous in a neighborhood of 0. Show that a particle in the neighborhood with total energy k and velocity directed toward 0 takes infinite time to reach 0.

The problem is to prove the divergence of the Integral (30) in this case $(a = 0)$. To prove this we obtain a test function of the form $\frac{A}{\sigma}$ and apply Theorem 10-6a and 10-6b. We use the tangent approximation to $k - U(\sigma)$, this time obtaining a lower bound for the error term in the form $C\sigma^2$. Apply the Law of the Mean twice to $U(\sigma) - k = U(\sigma) - [k - U'(0)\sigma]$ to obtain

$$\text{(i)} \qquad U(\sigma) - k = \sigma[U'(\sigma_1) - U'(0)]$$

$$= \sigma^2 U''(\sigma_2)$$

where σ_1 lies between σ and 0 and σ_2 between σ_1 and 0. From the continuity of U'' we can choose a neighborhood of 0 where $0 > U''(\sigma) > \frac{1}{2} U''(0)$ (Corollary 1 to Lemma 3-4). Insert this in (i) to obtain

$$k - U(\sigma) < \frac{1}{2}|U''(0)|\sigma^2 ;$$

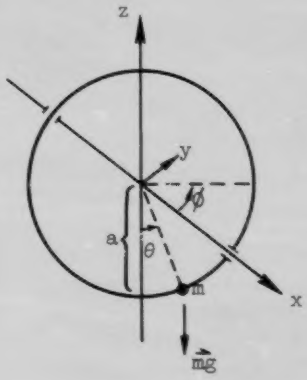
whence,

$$\frac{1}{\sqrt{k - U(\sigma)}} > \frac{1}{\sigma}\sqrt{\frac{2}{|U''(0)|}}$$

which proves the divergence of the integral.

13. Consider a particle of mass m which slides frictionlessly on a vertical circular hoop, where the hoop itself is spinning about its vertical diameter with constant angular speed ω. Describe the motion. (Hint: Use the energy conservation law in the Form (25) where s is arclength on the hoop).

Choose coordinates as shown in the figure, where $\emptyset$ is the angle the hoop makes with the xz-plane and θ the angle made by the position vector $\vec{X}$ of the particle with the downward vertical. For $\vec{X}$, obtain



$$\vec{X} = (a \cos \emptyset \sin \theta, a \sin \emptyset \sin \theta, -a \cos \theta)$$

where a is the radius of the hoop. Since $|\vec{X}|$ is the constant a we know that $\vec{v} = \frac{d\vec{X}}{dt}$ is perpendicular to $\vec{X}$ (Example 11-5a). The constant force N can only act normally to the hoop. To describe $\vec{N}$ introduce the unit tangent vector to the hoop

$$\vec{t} = (\cos \phi \cos \theta, \sin \phi \cos \theta, \sin \theta)$$

and the unit normal vector pointing away from the center of the hoop

$$\vec{n} = (\cos \phi \sin \theta, \sin \phi \sin \theta, -\cos \theta) .$$

For the third reference vector take the binormal $\vec{b} = \vec{t} \times \vec{n}$ given by

$$\vec{b} = (-\sin \phi, \cos \phi, 0) .$$

Thus $\vec{X} = a\vec{n}$.

The constraint force on the particle can only be exerted normally to the hoop; therefore it can be written in the form $\vec{N} = p\vec{n} + q\vec{b}$. The external force is that of gravity, $m\vec{g} = (0,0,-mg)$. In order to calculate $\frac{d^2\vec{X}}{dt^2}$ and apply Newton's Second Law, derive the relations

$$\vec{t}' = \omega \cos \theta \, \vec{b} - \theta'\vec{n} ,$$

$$\vec{n}' = \omega \sin \theta \, \vec{b} + \theta'\vec{t} ,$$

$$\vec{b}' = - \omega \cos \theta \vec{t} - \omega \sin \theta \, \vec{n}$$

where the prime denotes differentiation with respect to t , and we have used $\phi' = \omega$. From $\vec{X} = a\vec{n}$, then

$$\vec{X}' = a\theta'\vec{t} + a \omega \sin \theta \vec{b}$$

and

$$\vec{X}'' = (a\theta'' - a\omega^2 \sin \theta \cos \theta)\vec{t} - (a\theta'^2 + a\omega^2 \sin^2 \theta)\vec{n} .$$

From Newton's Second Law, however,

$$m\vec{X}'' = m\vec{g} + \vec{N}$$

$$= (-mg \sin \theta\vec{t} + mg \cos \theta\vec{n}) + (p\vec{n} + q\vec{b}) .$$

Since $\vec{X}''$ has no component in the direction of $\vec{b}$, it follows that $q = 0$. Consequently, on taking the dot product with $\vec{X}'$ we obtain for the kinetic energy T of the particle,

$$T' = m\vec{X}' \cdot \vec{X}'' = -mg \, a \sin \theta \, \theta' ;$$

whence,

$$T = \frac{m}{2}|\vec{X}'|^2 = mg \, a \cos \theta + K .$$

Thus energy conservation holds and the potential is exactly what it would be for the stationary hoop.

Finally, from the expression for $\vec{X}'$ above, we have

$$T = \frac{m}{2} a^2(\theta'^2 + \omega^2 \sin^2 \theta)$$

which yields the differential equation for θ

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{2g}{a}[k - \cos \theta] - \omega^2 \sin^2 \theta$$

with a new constant k . For $|k| < 1$ this motion is clearly the equation of a periodic oscillation which can be analyzed like that of the circular pendulum.

A student who wishes to analyze the possible motions of the system may be given the following hint. There is a value k for which θ is constant, provided ω is large enough. There exist small amplitude oscillations about this constant value of θ .

14. Consider a particle moving on a curve $\vec{X} = \vec{r}(s)$ subject to the conservation law (25) with a potential of the form

$$U(s) = A + Bs^2 + s^3F(s)$$

where $B > 0$ and the derivative F' exists and is bounded. Take $W(s) = A + Bs^2$ as an approximation to $U(s)$ near $s = 0$. Let the arclength for the motion under the potential U be given by $s = \phi(t)$, under the potential W , by $\sigma = \psi(t)$. Show for small amplitude oscillations that σ and s are close together for the half-cycle beginning with the initial states $\phi(0) = \psi(0) = \alpha > 0$, $\phi'(0) = \psi'(0) = 0$.

The initial condition prescribes equal amplitudes for the two motions, assuming $U(s)$ symmetric about $s = 0$.

From (25) we have for the two motions

(i) $$\left(\frac{ds}{dt}\right)^2 = b(\alpha^2 - s^2) + \alpha^3 f(\alpha) - s^3 f(s)$$

(ii) $$\left(\frac{d\sigma}{dt}\right)^2 = b(\alpha^2 - \sigma^2) ,$$

where $b = \frac{2B}{m}$ and $f(s) = \frac{2}{m} F(s)$. The motion (ii) is that of a simple harmonic oscillator

$$\sigma = \alpha \cos \sqrt{b}\, t .$$

To show that the motion (i) is approximately the same as (ii), use the Law of the Mean to estimate the extra term as follows,

$$(\frac{ds}{dt})^2 = b(\alpha^2 - s^2)[1 + g(s)]$$

where

$$g(s) = \frac{1}{b(\alpha + s)} \frac{\alpha^3 f(\alpha) - s^3 f(s)}{\alpha - s} \frac{D[s^3 f(s)]_{x=\xi}}{b(\alpha + s)}$$

where ξ is some number between α and s . Specifically,

$$g(s) \frac{3\xi^2 f(\xi) + \xi^3 f'(\xi)}{b(\alpha + s)}$$

where $0 \leq a < \xi < \alpha$. To make $|g(s)|$ less than some positive ϵ , observe that f' and f are both bounded (f is continuous), hence

$$\begin{aligned} |3\xi^2 f(\xi) + \xi^3 f'(\xi)| &\leq \xi^2(3|f(\xi)| + \xi|f'(\xi)|) \\ &\leq \alpha^2(3|f(\xi)| + \alpha|f'(\xi)|) \\ &\leq \alpha^2 M \end{aligned}$$

where M is a positive constant independent of ξ . Consequently

$$g(s) < \frac{\alpha^2 M}{b(\alpha + s)} \leq \frac{\alpha^2 M}{b\alpha} \leq \epsilon$$

provided we take $\alpha \leq \frac{\epsilon b}{M}$. Naturally we take α no smaller than we have to, and fix $\alpha = \frac{\epsilon b}{M}$. With $|g(s)| < \epsilon < 1$ we have

$$\sqrt{1 + g(s)} < \sqrt{1 + g(s) + \frac{g(s)^2}{4}} \leq 1 + \frac{\epsilon}{2}$$

and

$$\sqrt{1 + g(s)} \geq \sqrt{1 - |g(s)|} > \sqrt{1 - \epsilon} > 1 - \epsilon .$$

In either case, $|\sqrt{1 + g(s)} - 1| < \epsilon$. It follows that

$$\frac{1}{(1 + \epsilon)\sqrt{b(\alpha^2 - s^2)}} < \frac{1}{\sqrt{b(\alpha^2 - s^2)[1 + g(s)]}} < \frac{1}{(1 - \epsilon)\sqrt{b(\alpha^2 - s^2)}}$$

where the central quantity in the inequality is $-\frac{ds}{dt}$ from (i). Integrate and invert the relation between t and s to get

(iii) $\alpha \cos (1 + \epsilon)\sqrt{b}\, t < s < \alpha \cos (1 - \epsilon)\sqrt{b}\, t$.

Since cos is a continuous function it is clear that σ will approximate s within any given tolerance provided ϵ is small enough. An estimate of the difference is easily obtained by the Law of the Mean:

$$\alpha[\cos(1+\epsilon)\sqrt{b}\,t - \cos\sqrt{b}\,t] < s - \sigma < \alpha[\cos(1-\epsilon)\sqrt{b}\,t - \cos\sqrt{b}\,t]$$

whence

$$-\alpha\epsilon\sqrt{b}\,t\sin\xi_2 < s - \sigma < \alpha\epsilon\sqrt{b}\,t\sin\xi_1$$

where ξ_2 and ξ_1 define the appropriate means. It follows at once that

$$|s - \sigma| < \alpha\epsilon\sqrt{b}\,t < \frac{\alpha^2 M}{\sqrt{b}}\,t$$

where we use the value of α adopted earlier.

Solutions Exercises 12-4

1. Show that if $\vec{K} = \vec{0}$ in (2) then the trajectory is a straight line. What information can you obtain from (7) in this case?

From $\vec{X} \times \vec{X}' = \vec{0}$ we conclude that $\vec{X}$ and $\frac{d\vec{X}}{dt}$ are collinear. Thus all tangent lines to the trajectory pass through the origin. From the result of Example 11-5(i) it follows that the trajectory is a straight line through the origin. In this case the potential takes the simple form

$$\frac{m}{2}\rho'^2 + V(\rho) = E\ .$$

Consequently, if V is continuous, a particle in this case may reach the origin if its energy is great enough, unlike a particle subject to (7) with $K \neq 0$.

2. Use (7) or (8) to integrate the equations of motion for an inverse square force. (Hint: Replace ρ by $R = \frac{1}{\rho}$.)

Take $\phi(\rho) = -\frac{\alpha}{\rho^3}$ in (6) to obtain

$$V(\rho) = -\frac{\alpha}{\rho} + \text{constant.}$$

Now, with $R = \frac{1}{\rho}$, (8) becomes

$$\frac{mK^2}{2}\left(\frac{dR}{d\theta}\right)^2 = E + \alpha R - \frac{mK^2}{2} R^2\ ;$$

whence,

$$\frac{dR}{d\theta} = \pm\sqrt{C^2 - \left(R - \frac{\alpha}{mK^2}\right)^2}\ ,$$

where $C^2 = \frac{2E}{mK^2} - \frac{\alpha^4}{m^2K^4}$ and we may suppose $C > 0$. This equation has the solution

$$\theta - \theta_0 = \pm \arcsin \frac{1}{C}\left(R - \frac{\alpha}{mK^2}\right)$$

or

$$R = \frac{\alpha}{mK^2} \pm C \sin (\theta - \theta_0) .$$

This solution can be put in the Form (12) by taking $\beta = mK^2C$ and fixing θ_0 as $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ according to whether the sign is plus or minus.

3. Verify that the semi-minor axis b of the Ellipse (12) is given by $b = \dfrac{mK^2}{\sqrt{\alpha^2 - \beta^2}}$.

Put (12) in the form

$$\alpha\rho = mK^2 - \beta\rho \cos \theta$$

to obtain the cartesian equation

$$\alpha\sqrt{x^2 + y^2} = mK^2 - \beta x$$

whence,

$$x^2(\alpha^2 - \beta^2) + 2\beta mK^2x + \alpha^2y^2 = m^2K^4 .$$

In canonical form this becomes

$$\frac{\left[x = \dfrac{mK^2 \beta}{\alpha^2 - \beta^2}\right]^2}{\left[\dfrac{m^2K^4\alpha^2}{(\alpha^2 - \beta^2)^2}\right]} + \frac{y^2}{\left[\dfrac{m^2K^4}{(\alpha^2 - \beta^2)}\right]} = 1 ,$$

from which the result follows.

Alternatively, maximize $y = \rho \sin \theta$. Observe that $\frac{dy}{d\theta} = \frac{d}{d\theta}\left[\frac{mK^2 \sin \theta}{\alpha + \beta \cos \theta}\right]$ vanishes if and only if the numerator of the derivative vanishes; that is,

$$(\alpha + \beta \cos \theta)\cos \theta + \beta \sin^2 \theta = 0$$

or

$$\alpha \cos \theta + \beta = 0 .$$

Set $\cos \theta = - \frac{\beta}{\alpha}$ in the expression for y to obtain

$$y = \frac{mK^2\sqrt{1 - \frac{\beta^2}{\alpha^2}}}{\alpha - \frac{\beta^2}{\alpha}}$$

from which the result follows.

4. At what height will a satellite of the earth have the same period as the period of rotation of the earth about its axis? A synchronous satellite would be placed at this height. For the acceleration of gravity at the surface of the earth take $g = 980 \frac{\text{cm}}{\text{sec}^2}$ and for the radius of the earth, 6.37×10^8 cm .

It is implicit in the problem that the height is constant; hence the orbit is circular and the Equation (12) of the orbit reduces to $\rho = a$. The speed of motion $v = a\theta'$ is then given by (15),

$$v^2 = a^2\theta'^2 = \frac{\alpha}{ma} . \tag{ii}$$

To determine α use (17) to obtain

$$\alpha = GMm \tag{iii}$$

where m is the mass of the earth. Since the acceleration of gravity g is known at the surface of the earth use Newton's Second Law to get

$$mg = \frac{GMm}{c^2}$$

where c is the radius of the earth. Thus $GM = c^2 g$. Use this result in (iii) and (ii) to obtain

$$a^3 = \frac{c^2 g}{\theta'^2} .$$

Since 2π radians are covered at constant angular speed θ' in 24 hr. or 8.64×10^4 sec, we have $\theta' = \frac{2\pi}{8.64 \times 10^4}$ per sec. From the given data then,

$$a \approx 4.22 \times 10^9 \text{ cm.} ,$$

and the height above the surface of the earth is

$$a - c \approx 3.58 \times 10^9 \text{ cm.} ,$$

or approximately 22,000 miles.

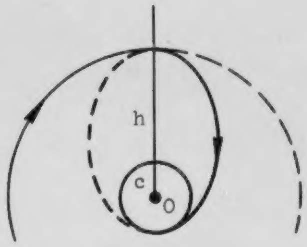
5. What is the escape velocity from the earth's surface?

From the text, the escape velocity is $\sqrt{\frac{2\alpha}{mc}}$ where c is the radius of the earth. From the solution of the preceding problem $\frac{\alpha}{m} = c^2 g$. Consequently the escape velocity is

$$2cg \approx 1 \cdot 12 \times 10^6 \ \frac{cm}{sec}$$

from the data given in Number 4, a speed of about 25,000 miles per hour.

6. Consider a satellite in a circular orbit about the earth. Its retrorockets are fired briefly so that its speed is reduced but its direction of motion and position are changed negligibly. Suppose the change of speed is just enough to bring the satellite to the earth's surface. Without air resistance, what must the change in speed be and how long does it take the satellite to reach the earth after the retrorockets are fired?



At the moment the satellite begins its descent its velocity is perpendicular to its position vector referred to the center of the earth (see figure). Therefore $\frac{d\rho}{d\theta} = 0$ and the satellite must be at one end of the major axis of its elliptical trajectory. Under the condition of the problem the other end of the major axis, the point of the orbit closest to the center of the earth, must be at the earth's surface. Thus the time of descent is half the period for the elliptical trajectory and is given by (16) as $T = \frac{\tau}{2} = \frac{\pi a^{3/2}}{2} \sqrt{\frac{m}{\alpha}}$. If the height of the circular orbit above the earth is h and the radius of the earth is c , then $a = \frac{h}{2} + c$. From (15) the velocity in the circular orbit is given by $v_c^{\,2} = \frac{\alpha}{m(h + c)}$ and at the initiation of the elliptical trajectory by

$$v_0^{\,2} = \frac{2\alpha}{m(h + c)} - \frac{\alpha}{m(\frac{h}{2} + c)} = \frac{2\alpha c}{m(h + c)(h + 2c)} \ .$$

From the solution to Number 4 take $\alpha = c^2 gm$ to obtain for the time of descent

$$T = \frac{\pi}{2} \, \frac{(\frac{h}{2} + c)^{3/2}}{c\sqrt{g}}$$

and the change in speed

$$v_c - v_0 = c\sqrt{\frac{g}{h + c}}\left[1 - \frac{1}{\sqrt{1 + \frac{h}{2c}}}\right].$$

If h is small relative to the radius c of the earth, as it is for manned orbital flights at this writing, then to a first approximation, $T = \frac{\pi}{2}\sqrt{\frac{c}{g}}$ and $v_c - v_0 = \frac{h}{4}\sqrt{\frac{g}{c}}$. Thus, with a slight correction for altitude, $T \approx 21$ min.

7. Prove if $\mu = 0$ in (27) the motion of the spherical pendulum reduces to that of the circular pendulum of Section 12-3.

If $\mu = 0$, then (27) implies $\theta' = 0$ unless $r = 0$ for all t. Consequently, the motion is confined to a vertical plane $\theta = \theta_0$.

8. Under what conditions is the maximum value z_2 equal to the minimum value z_1 of z for the motion of a spherical pendulum? Discuss the motion in this case.

In this case z is constant and the motion corresponds to a stable equilibrium for (29). This will occur when $-\frac{\mu^2}{\ell^2}$ is the minimum value of the "potential." In this case $r^2 = z^2 - \ell^2$ is constant. Hence, from (27), $\theta' = \frac{\mu}{r^2}$, the particle moves around the circle in the horizontal plane $z = z_1$ at constant speed.

9. What is the motion of the pendulum when $z = 0$? (Hint: Equation (29) is not convenient for the study of this motion).

From (24), angular momentum is conserved and the motion is planar. Set $\vec{X} \times \vec{X}' = \vec{K}$ and put $\vec{K} = (0,0,K)$, $\vec{X} = (\ell \cos\theta, \ell \sin\theta, 0)$. Then $\ell^2\theta' = K$. Thus θ' is constant and the particle moves on a great circle with constant speed.

10. Determine the magnitude of the force of constraint $\lambda\vec{X}$ in (22).

From Newton's equation of motion (22)

$$\lambda\vec{X} = m(\vec{g} - \vec{X}'') ,$$

whence,

$$\lambda\ell^2 = \lambda|\vec{X}|^2 = m(\vec{g}\cdot\vec{X} - \vec{X}''\cdot\vec{X})$$
$$= -mgz - m\frac{d}{dt}(\vec{X}\cdot\vec{X}') + m|\vec{X}'|^2 .$$

Since $|\vec{X}|^2$ is constant, $\vec{X}\cdot\vec{X}' = 0$ and this equation becomes

$$\lambda\ell^2 = -mgz + mv^2$$

where $v = |\vec{X}'|$. Consequently, for the constraint force $\vec{N} = \lambda\vec{X}$,

$$|\vec{N}| = \ell|\lambda| = \frac{m}{\ell}|v^2 - gz| .$$

Also, v^2 may be taken from (23), $v^2 = -2gz + 2k$ to give $|\vec{N}| = \frac{m}{\ell}|2k - 3gz|$.

Solutions Miscellaneous Exercises

1. Show that an object thrown with an upward component into a resisting atmosphere must come down to the same level at a speed less than its initial speed.

A force of resistance opposes the motion. Therefore it has the form

$$\vec{F}_r = -R\vec{X}'$$

where $R > 0$. The particular form of the resistance coefficient as a function of t , $\vec{X}$, $\vec{X}'$, or whatever, does not matter. From Newton's Second Law

$$m\vec{X}'' = m\vec{g} - R\vec{X}' . \tag{i}$$

Choose coordinates so that $\vec{g} = (0,0,-g)$. Take the dot product with $\vec{X}'$ in (i) and integrate from t_0 to t_1 , where t_1 is time of return, to get

$$\frac{m}{2}v_1^2 - \frac{mv_0^2}{2} = mg(z_0 - z_1) - \int_{t_0}^{t_1} Rv^2\,dt ,$$

where v_0 and v_1 are the initial and terminal speeds, respectively. Since $z_1 = z_0$ and the integrand is positive, the result follows at once.

2. (a) Prove if the angular momentum of particles is conserved then the force is a central force.

By hypothesis,

$$m(\vec{X} \times \vec{X}') = m\vec{K}$$

where $\vec{K}$ is a constant vector. Differentiate with respect to t to obtain

$$\vec{X} \times \vec{X}'' = \vec{0}$$

from which it follows that $\vec{X}''$ is collinear with $\vec{X}$. Hence the net force $\vec{F} = m\vec{X}''$ is a central force.

(b) Prove that a particle obeying Kepler's first two laws is subject to an inverse square force.

From the law that equal areas are swept out in equal times we know that angular momentum is conserved and by Part (a) the force is a central force. Since the path is an ellipse with a focus at the origin we have for the equation of the orbit in polar form

(i) $$\rho = \frac{1}{A + B \cos \theta} .$$

Since the motion is plane, take $\vec{X} = \rho(\cos \theta , \sin \theta)$ and obtain

$$\vec{X}' = \rho'(\cos \theta , \sin \theta) + \rho\theta'(-\sin \theta , \cos \theta) .$$

Since conservation of momentum holds, Equation (5) of Section 12-4 may be applied; thus take

(ii) $$\theta' = \frac{K}{\rho^2}$$

and obtain

$$\vec{X}' = \rho'(\cos \theta , \sin \theta) + \frac{K}{\rho}(-\sin \theta , \cos \theta) .$$

Now differentiate again to get

$$\vec{X}'' = (\rho'' - \frac{K}{\rho}\theta')(\cos \theta , \sin \theta) + (\rho'\theta' - \frac{K\rho'}{\rho^2})(-\sin \theta , \cos \theta) .$$

Apply (ii) again to find

$$\vec{X}'' = (\rho'' - \frac{K^2}{\rho^3})(\cos \theta , \sin \theta) .$$

It follows that the central force $\vec{F}$ is determined by λ where $\vec{F} = \frac{\lambda\vec{X}}{|\vec{X}|} = m\vec{X}''$, so that

(iii)
$$\lambda = m(\rho'' - \frac{K^2}{\rho^3}) .$$

In order to calculate $|\vec{F}|$ from (i) we must change the time derivative in (2) to a derivative with respect to θ . From (ii)

(iv)
$$\rho' = \theta' \frac{d\rho}{d\theta} = \frac{K}{\rho^2}\frac{d\rho}{d\theta} = -K \frac{d}{d\theta}(\frac{1}{\rho}) .$$

Differentiate again and apply (ii), as follows,

$$\rho'' = -K\theta' \frac{d^2}{d\theta^2}(\frac{1}{\rho}) = - \frac{K^2}{\rho^2}\frac{d^2}{d\theta^2}(\frac{1}{\rho}) .$$

Enter this result in (iii) to get

(v)
$$\lambda = - \frac{mK^2}{\rho^2}[\frac{d^2}{d\theta^2}(\frac{1}{\rho}) + \frac{1}{\rho}] .$$

Formula (v) may also be obtained by putting the expression (iv) for ρ' in Equation (7) of Section 12-4 and differentiating with respect to ρ . From Formula (5) we can calculate the central force given any orbit. In particular, for the orbit (i)

$$\lambda = - \frac{mK^2A}{\rho^2} ,$$

which is the inverse square law.

3. Show how the electric and magnetic field vectors $\vec{E}$ and $\vec{B}$ of Section 12-2(iii) must be modified to account for the motion of a charged particle in a coordinate frame which moves in translation with respect to the inertial frame with constant velocity $\vec{u}$.

Let the position vector of the particle in the moving frame be $\vec{X}^*$. Then

(i)
$$\frac{d\vec{X}^*}{dt} = \frac{dX}{dt} - \vec{u}$$

and, hence,

$$\frac{d^2\vec{X}^*}{dt^2} = \frac{d^2\vec{X}}{dt^2} .$$

Newton's Second Law in the moving frame is then obtained from (26) in the form

$$\text{(ii)} \qquad m\frac{d^2\vec{X}^*}{dt^2} = q[\vec{E} + (\frac{d\vec{X}^*}{dt} - \vec{u}) \times \vec{B}] .$$

Further, from (i)

$$\text{(iii)} \qquad \vec{X}^* = \vec{X} - t\vec{u} ,$$

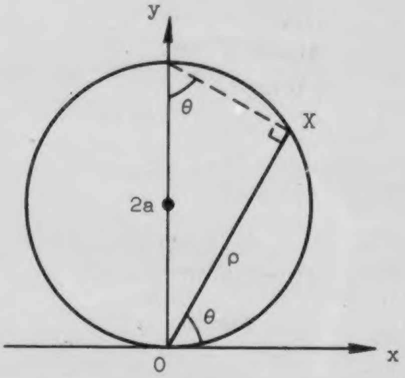
where the initial position of the particle $(t = 0)$ is taken at the origin in both the moving and the inertial frames. From (iii) we see that the motion may still be resolved into straight line and circular components with the vector $-t\vec{u}$ added to the linear component. Since the circular component of the motion is unaffected the magnitude and direction of the magnetic vector $\vec{B}$ should be the same in the moving frame as in the inertial frame. Consequently, the Lorentz force in the moving frame is expressed in the form of (i) by

$$q[\vec{E} + (\frac{d\vec{X}^*}{dt} - \vec{u}) \times \vec{B}] = q[\vec{E}^* + \frac{d\vec{X}^*}{dt} \times \vec{B}] ,$$

where the electric vector $\vec{E}^*$ and magnetic vector $\vec{B}^*$ are defined in the moving frame by

$$\vec{E}^* = \vec{E} - \vec{u} \times \vec{B} , \vec{B}^* = \vec{B} .$$

4. The path of an object attracted by a central force is a circle and the center of force is on the circle. Show that the force law is an inverse fifth power law, $\vec{F} = \frac{\alpha\vec{X}}{\rho^6}$ and show that the speed is proportional to $\frac{1}{\rho^2}$.



Choose coordinates as indicated in the accompanying figure. If the radius of the circle is a , then $\rho = a \sin \theta$, and

$$\begin{aligned}\vec{X} &= \rho(\cos \theta , \sin \theta) \\ &= a(\sin \theta \cos \theta , \sin^2 \theta) \\ &= \frac{a}{2}(\sin 2\theta , 1 - \cos 2\theta) .\end{aligned}$$

Now differentiate twice to obtain

(i) $$\vec{X}' = a\theta'(\cos 2\theta\,, \sin 2\theta)$$

$$\vec{X}'' = a(\theta'' \cos 2\theta - 2\theta'^2 \sin 2\theta\,, \theta'' \sin 2\theta + 2\theta'^2 \cos 2\theta) \ .$$

If $$\vec{F} = \mu \frac{\vec{X}}{|\vec{X}|}$$ then

$$\mu = m\vec{X}'' \cdot \frac{\vec{X}}{|\vec{X}|} = \frac{m\vec{X}'' \cdot \vec{X}}{a \sin \theta}$$

$$= \frac{ma}{\sin \theta}[\theta'' \sin 2\theta + 2\theta'^2(\cos 2\theta - 1)]$$

or

(ii) $$\mu = 2ma[\theta'' \cos \theta - 2\theta'^2 \sin \theta] \ .$$

From (5) of Section 12-4,

(iii) $$\theta' = \frac{K}{\rho^2} \ ;$$

whence

$$\theta'' = - \frac{2K}{\rho^3}\rho' = - \frac{2K}{\rho^3}\frac{d\rho}{d\theta}\theta' \ .$$

Since $\rho = a \sin \theta$, and with the use of (iii), get

(iv) $$\theta'' = - \frac{2aK^2}{\rho^5} \cos \theta \ .$$

Enter (iii) and (iv) in (ii) to get

$$\mu = -2m[\frac{2a^2K^2 \cos^2 \theta}{\rho^5} + \frac{2aK^2 \sin \theta}{\rho^4}] \ .$$

Since $A^2 \cos^2 \theta = a^2 - a^2 \sin^2 \theta = a^2 - \rho^2$ and $a \sin \theta = \rho$, this yields

$$\mu = - \frac{4ma^2K^2}{\rho^5} \ .$$

Alternatively, use (v) from the solution of Number 2(b) to get μ directly and so obtain the inverse fifth power force.

For the speed observe from (i) and (iii) that

$$v = |\vec{X}'| = 2a|\theta'| = \frac{2a|K|}{\rho^2}$$

which is the desired result.

5. (a) Let the path of a particle be given by $\vec{X} = \vec{r}(s)$ where s is arclength. Define the tangent $\vec{t}$ and principal normal $\vec{n}$ by $\vec{t} = \vec{r}'(s)$ and $\frac{d\vec{t}}{ds} = \kappa\vec{n}$ where the sign of the curvature κ is nonnegative (as in Exercises 11-6, No. 19). Let $v = \frac{ds}{dt}$ and $\alpha = \frac{dv}{dt}$ be respectively, the speed and acceleration along the curve. Show that the force on the particle is $\vec{F} = m\alpha\vec{t} + m\kappa v^2\,\vec{n}$.

For the velocity and acceleration vectors we have

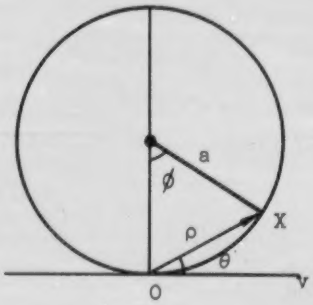
$$\frac{d\vec{X}}{dt} = \frac{ds}{dt}\,\frac{d\vec{X}}{ds} = v\vec{t}$$

and

$$\frac{d^2\vec{X}}{dt^2} = \frac{dv}{dt}\,\vec{t} + v\,\frac{ds}{dt}\,\frac{d\vec{t}}{ds} = \alpha\vec{t} + \kappa v^2\,\vec{n} ,$$

from which the result is immediate.

(b) Use the result of Part (a) to derive the inverse fifth power law in Number 4.



Introduce the central angle ϕ as a parameter. Thus $\phi = 2\theta$ where θ is the parameter in Number 4. We have from $s = a\phi$ $v = a\,\frac{d\theta}{dt} = 2a\,\frac{d\theta}{dt}$, $\alpha = a\,\frac{d^2\phi}{dt^2} = 2a\,\frac{d^2\theta}{dt^2}$. Use (5) of Section 12-4 to obtain, as in Number 4,

$$\frac{d\theta}{dt} = \frac{K}{\rho^2} , \quad \frac{d^2\theta}{dt^2} = -\frac{2K^2}{\rho^5}\cos\theta .$$

Then

$$v = \frac{2aK}{\rho^2} , \quad \alpha = -\frac{4a^2K^2}{\rho^5}\cos\theta .$$

Since $K = \frac{1}{a}$ and $\vec{t}\cdot\vec{n} = 0$, we have from $|\mu| = |F| = |mX''|$,

$$\mu^2 = m^2 \left[\frac{16a^2K^4}{\rho^8} + \frac{16a^4K^4 \cos \theta}{\rho^{10}} \right]$$

$$= \frac{16m^2a^2K^4}{\rho^{10}} [\rho^2 + a^2 \cos^2 \theta]$$

$$= \frac{16m^2a^4K^4}{\rho^{10}} ,$$

where at the last step put $\rho^2 = a^2 \sin^2 \theta$ in the bracket. The result agrees with Number 4.

6. Consider a rocket which is to lift a payload of mass M_1. Determine the amount of fuel relative to payload necessary to reach escape velocity in one minute from the earth given an exhaust velocity $v_c = 2 \times 10^5 \frac{\text{cm}}{\text{sec}}$ and a constant rate of fuel consumption. Neglect air resistance.

In the notation of Section 12-2(iv), the time to reach escape velocity is $t = \frac{m_1}{k} = 60$ sec. where $m_1 + M_1 = M_0$. From the solution of Exercises 12-2, Number 2(b), Equation (i), when the fuel is totally consumed the velocity is

$$\text{(i)} \qquad v = -gt + v_e \log \left(1 + \frac{m_1}{M_1}\right) .$$

Take the escape velocity $v = \sqrt{2cg} \approx 1.16 \times 10^6 \frac{\text{cm}}{\text{sec}}$ from the solution of Exercises 12-4, Number 5 to obtain

$$\frac{m_1}{M_1} = -1 + \exp\left(\frac{\sqrt{2cg} + gt}{v_e}\right) \approx 440 .$$

The data given are approximately those of the German V-2 rockets of World War II. The best that can be done to reduce the ratio of fuel to payload by shortening the time (take $t = 0$) yields a ratio of about 330. Clearly, the way to great improvement is to increase the exhaust velocity. For this the combustion chemistry of the propellent becomes an important factor.

Teacher's Commentary

Chapter 13

NUMERICAL ANALYSIS

TC13-1. Introduction.

The purpose of this chapter is not numerical computation as such, but analysis of algorithms or methods of computation. The ability to compose algorithms for electronic computation is a skill in great demand nowadays. As the mathematical, scientific, and technological questions we ask of computers probe more deeply, the standard routines of the programmer become less adequate and analytical skill plays a much larger role. In sparing us the mental effort of mechanical arithmetic electronic computation has given us more time to devote to non-routine analytical questions, but it has also given rise to a wealth of new analytical problems.

Time sharing systems make computers available to a growing number of students (sometimes as early as sixth grade) for their lengthy calculations. By writing their own programs, these students are developing analytical insights formerly undeveloped until the undergraduate years.

The central section of this chapter is Section 13-3, Taylor's Theorem with Remainder. It sharpens ideas developed earlier in the text and it is invaluable for future applications. Taylor's Theorem is undoubtedly the single most important method of error estimation.

Many of the numerical examples and exercises analyzed in this chapter stem from earlier discussions (e.g., compare Section 8-6 (13) with Section 13-4 (16), or Exercises 6-M, No. 11 with Section 13-4 (9)).

Solutions Exercises 13-1

1. (a) Obtain a method for computing the square root of a positive number to within any prescribed tolerance ϵ . Obtain estimates to determine at what stage the process may be brought to an end.

   (b) Use the given method to obtain $\sqrt{7}$ accurate to 5 significant figures.

13-1

There is no end of possible answers to this question. Here we give two sophisticated methods, first the scheme commonly taught in arithmetic courses, next an "original" scheme. This problem illustrates what may be involved in calculating values of the square roots for the evaluation of the integral I in Equation (1).

Let s_j be the "best" lower estimate to $\sqrt{x}$ to $j+1$ significant figures; that is, s_j has the decimal representation

$$s_j = a_0 10^k + a_1 10^{k-1} + \dots + a_j 10^{k-j} ,$$

where $a_0 \neq 0$, and s_j must satisfy the condition

$$s_j^2 \le x < (s_j + 10^{k-j})^2 .$$

Now suppose we wish to determine the best lower estimate to one more place. We require

$$(s_{j+1})^2 = (s_j + a_{j+1} 10^{k-j-1})^2 \le x .$$

Hence for $r_j = x - s_j^2$ we must have

(i) $$2a_{j+1} s_j 10^{k-j-1} + a_{j+1}^2 100^{k-j-1} \le r_j .$$

Thus a_{j+1} must satisfy

$$2a_{j+1} s_j 10^{k-j-1} \le r_j$$

or

(ii) $$a_{j+1} \le \frac{r_j}{2s_j} .$$

In the light of (ii) we try $b_{j+1} = \left[\frac{r_j}{2s_j} \right]$ as a candidate for a_{j+1}. If this figure is too large to satisfy (i) we try the digits in descending order from b_{j+1} until we reach one which satisfies (i).

The error $\sqrt{x} - s_j$ is less than 10^{k-j} and we need only take enough steps to make $10^{k-j} < \epsilon$, or $j < k + \log_{10}(\frac{1}{\epsilon})$.

We exhibit the usual way of laying out the computation for $\sqrt{7}$. Redundant zero digits are omitted. It is easily verified if x is written in base 100 that s_j represents the best lower decimal approximation to $\sqrt{x_j}$ where x_j is taken as x to five significant figures in base 100 . Thus the computation is arranged by grouping the digits of x in pairs as though x were written to base 100 . In the following labeled computation the symbol $\{a_0a_1a_2 \dots a_k\}$ represents decimal place numeration, not a product.

	7 00 00 00 00	
$a_0 = 2$	4	a_0^2
	3 00	r_0 , $b_1 = \left[\!\!\left[\frac{30}{4}\right]\!\!\right] = 7$
	3 29	$[20a_0 + b_1]b_1$, $b_1 > a_1$ try again
	3 00	r_0 , $b_1^* = 6$
$a_1 = 6$	2 76	$[20a_0 + b_1^*]b_1^*$
	24 00	r_1 , $b_2 = \left[\!\!\left[\frac{240}{52}\right]\!\!\right] = 4$
$a_2 = 4$	20 96	$[20\{a_0a_1\} + b_2]b_2$
	3 04 00	r_2 , $b_3 = \left[\!\!\left[\frac{3040}{528}\right]\!\!\right] = 5$
$a_3 = 5$	2 64 25	$[20\{a_0a_1a_2\} + b_3]b_3$
	39 75 00	r_3 , $b_4 = \left[\!\!\left[\frac{39750}{5290}\right]\!\!\right] = 7$
$a_4 = 7$	37 06 49	$[20\{a_0a_1a_2a_3\} + b_4]b_4$.

This yields $s_5 = 2.6457$.

For the second method, let a_0 be the first signigicant digit of $\sqrt{x}$ as above. Then $1 \leq a_0 \leq \frac{x}{10^{2k}} < 1 + a_0$. Set $A = \frac{x}{10^{2k}}$ for simplicity. It is clearly sufficient for the approximation to $\sqrt{A}$, since we need only multiply by 10^k , to approximate $\sqrt{x}$ to the same number of significant figures. Take whichever integer a_0 or $1 + a_0$ which has the closer square to A and denote it by z_1 . Set

$$\alpha_1 = |z_1 - \sqrt{A}|$$

where for the absolute error we have $\alpha_1 < 1$. Now form the k-th power

$$\begin{aligned}(z_1 - \sqrt{A})^k &= z_1^{\,k} - \binom{k}{1}\sqrt{A}\, z_1^{k-1} + \binom{k}{2}A z_1^{k-2} - \dots \\ &= \sum_{r=0}^{\mu} \binom{k}{2r} A^r z_1^{k-2r} - \sqrt{A} \sum_{r=0}^{\nu} \binom{k}{2r+1} A^r z_1^{k-2r-1} \\ &= p_k - q_k\sqrt{A} ,\end{aligned}$$

where $\mu = \left[\frac{k}{2}\right]$, $\nu = \left[\frac{k-1}{2}\right]$. Now set $z_k = \frac{p_k}{q_k}$. We have

$$\text{(i)} \qquad \alpha_k = |z_k - \sqrt{A}| = \frac{|z_1 - \sqrt{A}|^k}{q_k} = \frac{\alpha_1^k}{q_k} .$$

Note that under the conditions on A and z_1, $\sqrt{A} \geq 1$ and $z_1 \geq 1$. Since the binomial coefficients are natural numbers, hence no less than 1, it follows that $q_k \geq 1$. Since $\alpha_1 < 1$ it follows from (i) that the error can be made less than any given tolerance by taking k sufficiently large.

We would like a rough estimate for the absolute error α_k so that we may determine k. It is important to realize that the effort of justifying an error estimate rigorously may be disproportionate to the end achieved. Here we use a nonrigorous approach, but a rigorous estimate can be obtained so the initial lack of rigor will not weaken the conclusion. Suppose z_1 is a reasonably good approximation to $\sqrt{A}$ so that

$$\begin{aligned}q_k &= \sum_{r=0}^{\nu} \binom{k}{2r+1} A^r z_1^{k-2r-1} \\ &\approx \sum_{r=0}^{\nu} \binom{k}{2r+1} (\sqrt{A})^{2r} (\sqrt{A})^{k-2r-1} \\ &\approx (\sqrt{A})^{k-1} \sum_{r=0}^{\nu} \binom{k}{2r+1} .\end{aligned}$$

The sum in this expression is the sum of every other binomial coefficient so we guess that it is approximately half the total of all the binomial coefficients, that is, about $\frac{2^k}{2}$ (this is actually the exact value of the sum). Thus we obtain $q_k \approx (2\sqrt{A})^{k-1}$ and hence from (i),

$$\alpha_k = \frac{\alpha_1^{\ k}}{(2\sqrt{A})^{k-1}} \ . \tag{ii}$$

Now let us apply the method to calculate $\sqrt{7}$. Take $z_1 = 3$. From

$$z_1^{\ 2} - 7 = (z_1 - \sqrt{7})(z_1 + \sqrt{7}) = 2 \ .$$

Consequently, $\alpha_1 = 3 - \sqrt{7} = \frac{2}{3 + \sqrt{7}}$ and since $\sqrt{7} > 2$,

$$\alpha_1 < \frac{2}{5} \ . \tag{iii}$$

Note also from $\sqrt{7} = 3 - \alpha_1$ that

$$\sqrt{7} > 2.6 > \frac{5}{2} \ . \tag{iv}$$

Insert the inequalities (iii) and (iv) in (ii) to get the conjectured inequality

$$\alpha_k < (\tfrac{2}{5})^k(\tfrac{1}{5})^{k-1} \leq \frac{2^k}{5^{2k-1}} \ .$$

We wish to make α_k less than 10^{-4} and from our conjecture this will occur if k is sufficiently large to make $\frac{2^k}{5^{2k-1}} < \frac{1}{10^4}$. This last inequality holds first for $k = 5$. Now, we obtain p_k and q_k from

$$p_5 - q_5\sqrt{7} = (3 - \sqrt{7})^5 \ ,$$

namely,

$$p_5 = 3^5 + 10 \times 7 \times 3^3 + 5 \times 7^2 \times 3 = 2868$$

$$q_5 = 5 \times 3^4 + 10 \times 7 \times 3^2 + 7^2 = 1084 \ .$$

From this result and (iii) we obtain for the absolute error in (i),

$$\alpha_5 < (\tfrac{2}{5})^5 \ \frac{1}{1084} < (\tfrac{1}{2})^5 \ \frac{1}{1000} < \frac{1}{32000} \ .$$

It follows that

$$z_5 = \frac{p_5}{q_5} = 2.6457 \ \ldots$$

approximates $\sqrt{7}$ accurately to five significant figures.

2. The idea of using Riemann sums to approximate the integral I given by Equation (1) has led us into formal complications. In such an approximation, the integrand is approximated by piecewise constant functions. Use a little more ingenuity in approximating the integrand and obtain upper and lower estimates for I . (Hint: note that $\int_x^{\pi/2} \frac{d\psi}{\sqrt{\cos\psi}} = \int_0^{\pi/2-x} \frac{d\psi}{\sqrt{\sin\psi}}$ and use estimates for cos and sin obtained from Example 7-5(b).)

Observe that

$$\text{(i)} \qquad I = \int_0^{\pi/4} \frac{d\psi}{\sqrt{\cos\psi}} + \int_0^{\pi/4} \frac{d\psi}{\sqrt{\sin\psi}}$$

and estimate the two integrals separately. Since $0 \le \psi \le \frac{\pi}{4} < 1$ we have from the results of Example 7-5b,

$$\cos\psi \le 1 - \frac{\psi^2}{2} + \frac{\psi^4}{24} \le 1 - \frac{\psi^2}{2} + \frac{\psi^2}{24} \le 1 - \frac{11}{24}\psi^2 .$$

Coupling this with the lower estimate for $\cos\psi$, we have second degree estimates for $\cos\psi$.

$$\text{(ii)} \qquad 1 - \frac{\psi^2}{2} \le \cos\psi \le 1 - \frac{11}{24}\psi^2 .$$

Similarly, with

$$\sin\psi \ge \psi - \frac{\psi^3}{6} \ge \psi - \frac{\psi}{6} \ge \frac{5}{6}\psi ,$$

we obtain first degree estimates for $\sin\psi$:

$$\text{(iii)} \qquad \frac{5}{6}\psi \le \sin\psi \le \psi .$$

Next, integrate from 0 to $\frac{\pi}{4}$ in (ii) to obtain the estimates for the second integral in (i),

$$\sqrt{\frac{24}{11}} \arcsin \frac{\pi}{4}\sqrt{\frac{11}{24}} \le \int_0^{\pi/4} \frac{d\psi}{\sqrt{\cos\psi}} \le \sqrt{2} \arcsin \frac{\pi}{4\sqrt{2}} .$$

Sacrifice a little in this inequality by using the inequality for arcsin derived from (iii)

$$x \le \arcsin x \le \frac{6x}{5}$$

to obtain

(iv) $$\frac{\pi}{4} \leq \int_0^{\pi/4} \frac{d\psi}{\sqrt{\cos\,\psi}} \leq \frac{3\pi}{10}\ .$$

For the second integral in (i) obtain on integrating from 0 to $\frac{\pi}{4}$ in (iii),

(v) $$\sqrt{\pi} \leq 2\sqrt{\frac{\pi}{4}} \leq \int_0^{\pi/4} \frac{d\psi}{\sqrt{\sin\,\psi}} \leq 2\sqrt{\frac{\pi}{4}}\,\sqrt{\frac{6}{5}} \leq \sqrt{\frac{6\pi}{5}}\ .$$

Add (iv) and (v) to get

$$2.5 < \frac{\pi}{4} + \sqrt{\pi} \leq I \leq \frac{3\pi}{10} + \sqrt{\frac{6x}{5}} < 2.9\ .$$

TC13-2. Iteration.

A geometrical picture of an iteration scheme may help make the idea more intuitive.

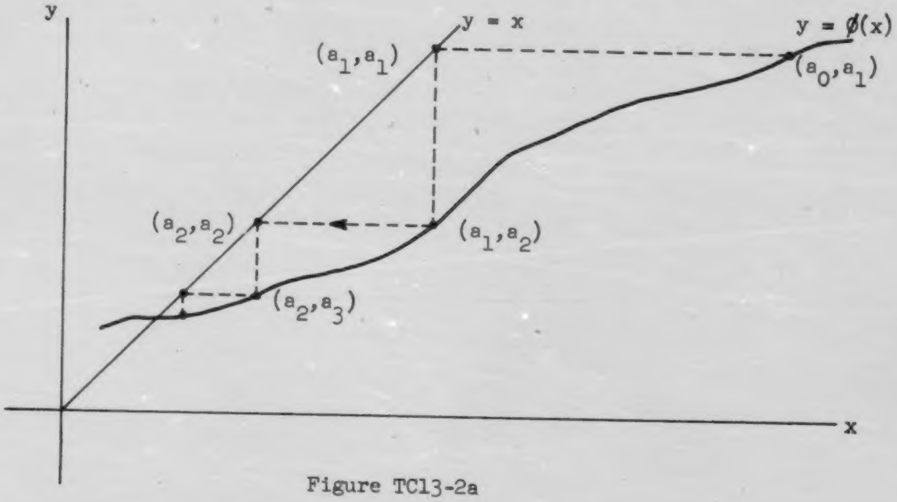

y
y = x
y = $\phi(x)$
(a_1,a_1)
(a_0,a_1)
(a_2,a_2)
(a_1,a_2)
(a_2,a_3)
x

Figure TC13-2a

Draw the graphs $y = \phi(x)$ and $y = x$, (Figure TC13-2a). From the point (a_0,a_1) where $a_1 = \phi(a_0)$ proceed horizontally to the graph $y = x$, thence vertically to the graph $y = \phi(x)$ and repeat the procedure as indicated in the figure. For an alternating scheme (Exercises 13-2, No. 2) the geometrical procedure follows a spiral (Figure TC13-2b)

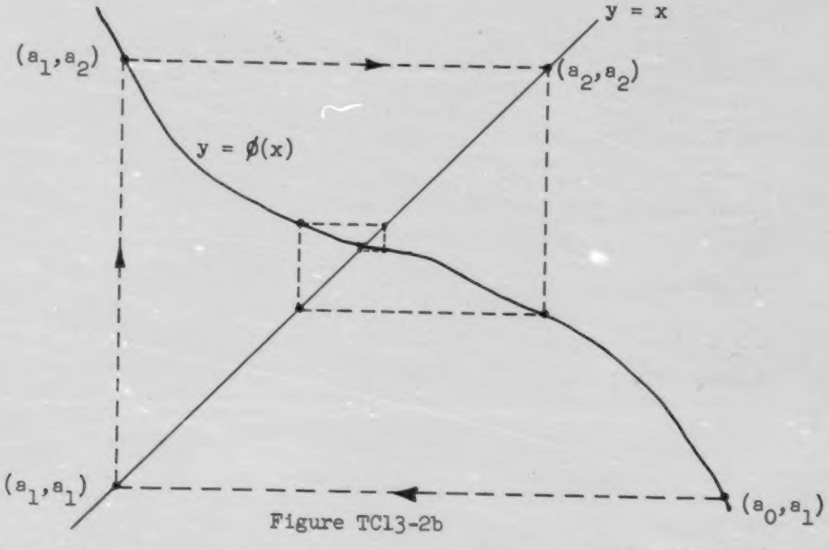

y = x
(a_1,a_2)
(a_2,a_2)
y = $\phi(x)$
(a_1,a_1)
(a_0,a_1)

Figure TC13-2b

The proof of Theorem 13-2 requires the Least Upper Bound Principle (Appendix A1-5).

Solutions Exercises 13-2

1. Devise an iteration scheme to approximate $\sqrt{A}$ for any positive A. Show how to choose an initial estimate for $\sqrt{A}$ so that the scheme converges and verify that the error can be brought below any given tolerance.

Use Newton's Method (14) for the solution of $f(x) = x^2 - A = 0$, namely

$$\begin{aligned} a_{k+1} &= a_k - \frac{1}{2a_k}(a_k^2 - A) \\ &= \frac{1}{2}\left(a_k + \frac{A}{a_k}\right) . \end{aligned}$$

If $e_k = a_k - \sqrt{A}$ then

$$\begin{aligned} e_{k+1} = a_{k+1} - \sqrt{A} &= \frac{1}{2}\left(a_k + \frac{A}{a_k}\right) - \sqrt{A} \\ &= \frac{1}{2}\left(e_k + \sqrt{A} + \frac{A}{e_k + \sqrt{A}}\right) - \sqrt{A} \\ &= \frac{1}{2}\left(e_k + \sqrt{A} + \frac{\sqrt{A}(e_k + \sqrt{A}) - e_k\sqrt{A}}{e_k + \sqrt{A}}\right) - \sqrt{A} \\ &= \frac{e_k^2}{2(e_k + \sqrt{A})} . \end{aligned}$$

If $e_k > 0$ it follows that $0 < e_{k+1} < \frac{e_k}{2}$; thus the error is cut at least in half at each stage. To guarantee convergence it is therefore sufficient to take $a_0 > \sqrt{A}$. The scheme is then monotone, that is, the successive iterants decrease. Now, for a more accurate idea of the rate of convergence, suppose that a_k is an upper estimate for $\sqrt{A}$ accurate to n decimal places and $1 \leq A$ (remember that a shift in decimal point in $\sqrt{A}$ amounts to the replacement of A by $100^P A$ so the inequality $A \geq 1$ is not an important restriction). In that case $0 \leq e_k < \frac{10^n}{2}$. Consequently,

$$e_{k+1} = \frac{e_k^2}{2(e_k + \sqrt{A})} \leq \frac{e_k^2}{2\sqrt{A}} \leq \frac{e_k^2}{2} < \frac{10^{2n}}{4} .$$

Thus the number of accurate decimal places is at least doubled at each iteration.

2. (a) An iteration scheme is called alternating if the error changes sign at each iteration. Since any two consecutive iterants approximate the solution from above and below, the value of an alternating scheme is that it permits an estimate of the error without a separate error analysis. Find a sufficient condition that a convergent iteration scheme be alternating.

Let $a = \phi(a)$ be the solution of (3) to be approximated. If the iteration scheme (4) is alternating then $\frac{e_{k+1}}{e_k} < 0$. But,

$$\frac{e_{k+1}}{e_k} = \frac{a_{k+1} - a}{a_k - a} = \frac{\phi(a_k) - \phi(a)}{a_k - a} = \phi'(\xi_k)$$

where ξ_k lies between a and a_k . Thus if ϕ has a negative derivative in some neighborhood of a the scheme is alternating if the iterants lie in the given neighborhood.

(b) Construct a convergent alternating scheme for calculating $\sqrt{A}$.

Consider f in (3) of the form $f : x \to x^2 - A$ to obtain

$$\phi(x) = x + c(x^2 - A) .$$

In order to guarantee convergence in a neighborhood of $\sqrt{A}$ require $-1 < \phi'(\sqrt{A}) < 1$, and to make the scheme alternating, require $\phi'(\sqrt{A}) < 0$. From the two conditions on $\phi'(\sqrt{A})$ obtain

$$-1 < 1 + 2c\sqrt{A} < 0$$

or

(i) $$-\frac{1}{\sqrt{A}} < c < -\frac{1}{2\sqrt{A}} .$$

(c) Use the alternating scheme obtained in Part (b) to calculate $\sqrt{3}$ accurately to two decimal places.

From (i) we wish to choose c so as to satisfy

$$\frac{1}{12} < c^2 < \frac{1}{3} .$$

For rapidity of convergence we would like c^2 to be as close to the lower value as convenient. Thus we choose $c^2 = \frac{1}{9}$ and $c = -\frac{1}{3}$. For ϕ in (5) given by $f : x \to x^2 - 3$, and $g(x) = c$, the iteration scheme is then

$$\phi(x) = x - \frac{x^2}{3} + 1 .$$

Observe that $\phi'(x) = 1 - \frac{2x}{3} \leq 0$ for $x \geq \frac{3}{2}$. Take the initial approximation as $a_0 = \frac{3}{2}$. From $a_{k+1} = a_k - \frac{a_k^2}{3} + 1$ obtain successively $a_0 = \frac{3}{2} = 1.5$, $a_1 = \frac{7}{4} = 1.75$, $a_2 = \frac{83}{48} = 1.729 \ldots$, $a_4 = 1.732 \ldots$. Since $\sqrt{3}$ lies between a_2 and a_4 we conclude that $\sqrt{3} \approx 1.73$ accurately to two decimal places.

3. Obtain an iteration scheme for $\sqrt[n]{A}$, demonstrate convergence, and estimate the error in the k-th iterant.

Use Newton's Method for $f(x) = x^n - A = 0$. Then

$$\phi(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^n - A}{nx^{n-1}}$$

$$= (1 - \frac{1}{n})x + \frac{A}{nx^{n-1}} .$$

Set $a = \sqrt[n]{A}$. Then

$$\text{(i)} \qquad e_{k+1} = a_{k+1} - a = e_k - \frac{(a_k^n - a^n)}{na_k^{n-1}} .$$

Now, from the tangent approximation, Section 5-7, we have

$$\text{(ii)} \qquad a^n = a_k^n + na_k^{n-1}(a - a_k) + \epsilon_k$$

where $|\epsilon_k| < M_2(a - a_k)^2$, M_2 being an upper bound on $\frac{d^2}{dx^2}(x^n)$ for x between a and a_k . Enter (ii) in (i) to obtain

$$\text{(iii)} \quad |e_{k+1}| = \frac{\epsilon_k}{na^{n-1}} \leq \frac{n(n-1)\xi^{n-2}}{na^{n-1}} e_k^2 \leq \frac{(n-1)\xi^{n-2}}{a^{n-1}} e_k^2$$

for $\xi = \max(a_k, a)$, (we may assume $a_k > 0$ since all iterants will be positive if $a_0 > 0$). Since we expect $\xi \approx a$ the bound on $|e_{k+1}|$ given by (iii) should be approximately $\frac{n-1}{a} e_k^2$ and this is usually good enough for practical purposes.

4. (a) Prove the following theorem. Suppose $f(a) = 0$. If f has two continuous derivatives on a neighborhood of a and if $f'(a) \neq 0$, then Newton's Method (14) converges if the initial estimate is sufficiently close to a. (Hint: use the Law of the Mean twice to approximate $f(a_k)$ as for the tangent approximation, Section 5-7).

   (b) Show that Newton's Method is monotone (the error has constant sign) if in addition to the conditions of Part (a), $f''(a) \neq 0$.

Since $f(a) = 0$, it follows from the Law of the Mean that

$$\begin{aligned} f(a_k) &= f(a_k) - f(a) = f'(u)(a_k - a) \\ &= f'(a_k)(a_k - a) + [f'(u) - f'(a_k)](a_k - a) \\ &= f'(a_k)(a_k - a) - f''(v)(a_k - u)(a_k - a) \end{aligned}$$

where u lies between a and a_k and v lies between u and a_k. Consequently, from (14),

$$\begin{aligned} e_{k+1} &= a_{k+1} - a = a_k - a - \frac{f(a_k)}{f'(a_k)} \\ &= e_k - \frac{f'(a_k)e_k - f''(v)(a_k - u)e_k}{f'(a_k)}, \end{aligned}$$

hence

$$e_{k+1} = \frac{f''(v)}{f'(a_k)} e_k(a_k - u) . \tag{i}$$

Now, since $f'(a) \neq 0$ there is a neighborhood of a where $|f'(x)| > \frac{|f(a)|}{2}$. Furthermore, since f'' is continuous on a neighborhood of a, there is a neighborhood where f'' is bounded, $|f''(x)| < M$. Furthermore $|a_k - u| < |e_k|$. Let δ be the radius of the smaller of these neighborhoods. It follows, for a_k within the δ-neighborhood of a, that

$$|e_{k+1}| < Ke_k^2$$

where $K = \frac{2M}{f'(a)}$. For convergence we need only guarantee that $\left|\frac{e_{k+1}}{e_k}\right| < r < 1$ for some constant r. For this it is sufficient to choose a_0 so that $|e_0| < \mathrm{Min}\{\frac{1}{K}, \delta\}$.

To prove Part (b) observe that $\operatorname{sgn}(a_k - u) = \operatorname{sgn}(a_k - a) = \operatorname{sgn} e_k$. Consequently, $\operatorname{sgn} e_{k+1} = \operatorname{sgn} \frac{f''(v)}{f'(a_k)} = \operatorname{sgn} \frac{f''(a)}{f'(a)}$ for a_k sufficiently close to a . Thus the sign of the error is constant once it falls below some sufficiently small bound.

After we have proved Taylor's Theorem (Section 13-3) it will be possible to relax the conditions $f'(a) \neq 0$ and $f''(a) \neq 0$ employed in this exercise.

5. Obtain the greatest zero of f to 3 decimal places, where
(a) $f(x) = x^6 + 6x - 9$;

Since $f(1) = -2$, $f(2) = 51$ there is a root a in the open interval $(1,2)$. Furthermore, since f is increasing for $x > 0$ the root is unique and it is the largest root. Use

$$\begin{aligned} f'(a) &= 6a^5 + 6 = 6\left(\frac{a^6}{a} + 1\right) \\ &= 6\left(\frac{9 - 6a}{a} + 1\right) \\ &= \frac{6}{a}(9 - 5a) . \end{aligned}$$

Since we expect $a \approx 1$ we suppose $f'(a) \approx 24$ and take $c = -\frac{1}{24}$ in (11):

$$\begin{aligned} a_{k+1} &= a_k - \frac{1}{24}(a_k^6 + 6a_k - 9) \\ &= \frac{9 + 18a_k - a_k^6}{24} . \end{aligned}$$

Take $a_0 = 1$ and obtain, successively, $a_1 \approx 1.08 \ldots$, $a_2 \approx 1.12$, $a_3 \approx 1.13$. At this point it may be worthwhile to re-estimate $f'(a)$ using $a \approx 1.13$. With this approximation, obtain

$$f'(a) = \frac{6}{a}(9 - 5a) \approx 18$$

and $c = -\frac{1}{18}$. Then use

$$a_{k+1} = \frac{9 + 12a_k - a_k^6}{18} , \qquad (k \geq 3)$$

for the next approximations, $a_3 \approx 1.13$, $a_4 \approx 1.1378$, $a_5 \approx 1.1380$.

To conclude the computation try an expedient. The successive iterants are increasing. We expect that 1.138 is close to the desired accuracy. To check this, on the assumption that the iteration scheme is monotone for $a_k < a$, try $a_6 = 1.1385$. Then $a_7 \approx 1.1380 < a_6$. Thus 1.1385 is presumably an overestimate and, to three decimal places, $a = 1.138$. The assumptions made above can be verified, as some students may wish to do. The point, however, is that in any practical situation a balance has to be established between the need for confidence in a result and the effort required for proof. Empirically obtained answers should not be disdained in appropriate situations.

(b) $f(x) = x^3 - 4x^2 + 4x - 2$;

Since $f(2) = -2$, $f(3) = 1$ and f is increasing for $x \geq 2$ (from $f'(x) = (3x - 2)(x - 2)$) it follows that the largest root a is the one root in the open interval $(2,3)$, probably with a closer to 3 than to 2. Use Newton's Method,

$$a_{k+1} = a_k - \frac{a_k^3 - 4a_k^2 + 4a_k - 2}{3a_k^2 - 8a_k + 4},$$

with $a_0 = 3$. Then $a_1 = 3 - \frac{1}{7} \approx 2.86$, $a_2 \approx 2.84$, $a_3 \approx 2.8396$, $a_4 = 2.8394$. To conclude the process as in Part (a) try $a_5 = 2.839$ and obtain $a_6 = 2.8392 > a_5$. To three decimal places we have $a_5 = 2.839$.

(c) $f(x) = \sum_{n=0}^{15} x^n = 48$.

Follow the method of Example 13-2b. The desired root lies in $(1,2)$, presumably closer to 1. With $x = 1 + u$ obtain for small z

(1) $f(x) = \frac{x^{16} - 1}{x - 1} = \frac{(1 + u)^{16} - 1}{u} \approx 16 + 120u + 560u^2$.

Thus as an initial estimate for the solution take u satisfying $16 + 120u + 560u^2 + \dots = 48$. Ignore terms higher than second degree to obtain as a first estimate $\frac{1}{7} < u_0 < \frac{1}{6}$. Take $u_0 = \frac{1}{7}$

and $a_0 = \frac{8}{7}$. Multiply in $f(x) = 0$ by $x - 1$ to obtain

$$F(x) = x^{16} - 48x + 47 = 0$$

and

$$\begin{aligned} F'(a) &= 16a^{15} - 48 \\ &= 16(\frac{a^{16}}{a} - 3) \\ &= 16(45 - \frac{47}{a}) . \end{aligned}$$

For the iteration scheme, take

$$a_{k+1} = a_k - \frac{a_k^{16} - 48a_k + 47}{16(45 - \frac{47}{a_k})} .$$

Obtain successively, $a_0 = 1.14$, $a_1 = 1.134$, $a_2 = 1.134$ so that no further improvement is obtained at this level of accuracy.

6. Show if the iteration scheme (4) converges to a number a in the domain of ϕ , and if a is a point of continuity of ϕ , that a is a solution of (3); that is, $\phi(a) = a$.

From the continuity of ϕ we know that for any error tolerance ϵ , we may ensure $|\phi(x) - \phi(a)| < \epsilon$ by requiring $|x - a| < \delta$ for some positive δ . Since the iteration scheme converges we also have $|a_k - a| < \epsilon^*$ for all sufficiently large k , say $k > M$. Choose $\epsilon^* = \text{Min}\{\epsilon,\delta\}$. Then $|a_{k+1} - a| < \delta$ whence, by the continuity condition, $|\phi(a_k) - \phi(a)| < \epsilon$, but, by the convergence condition,

$$|a_{k+1} - a| = |\phi(a_k) - a| < \epsilon .$$

It follows that

$$\begin{aligned} |\phi(a) - a| &\le |\phi(a) - \phi(a_k)| + |\phi(a_k) - a| \\ &< 2\epsilon . \end{aligned}$$

Thus $|\phi(a) - a|$ is less than any positive number, hence must be zero.

7. Verify the error estimate (17) in the Picard iteration scheme for a separable equation under the conditions of Theorem 13-2.

For simplicity, fix $r = \frac{1}{2}$ in (17). Let Q be an upper bound for $g'(y)$ on a neighborhood of y_0, say $|g'(y)| < Q$ for $|y - y_0| < \beta$. Choose α so that the solution u (which is continuous since it is differentiable) differs from y_0 by less than β for $|x - x_0| \leq \alpha$ and let P be the maximum of $|f(x)|$ when $|x - x_0| \leq \alpha$. Set $M = PQ$. Then for $|x - x_0| < \alpha$ we have $|u(x) - y_0| < \beta$ and we may take $\epsilon_0 = \beta$. For the first iterant we have

$$e_1(x) = u_1(x) - u(x) = \int_{x_0}^{x} f(\xi)[g(y_0) - g(u(\xi))]d\xi$$

$$= \int_{x_0}^{x} f(x)g'(\eta_1)[y_0 - u(\xi)]d\xi .$$

Hence, for $|x - x_0| < \alpha$

$$|e_1(x)| \leq |x - x_0|M\beta .$$

Now we may replace α above by any smaller positive number without violating any condition already required. In particular, we may suppose $\alpha < \frac{1}{2M}$. Then $|e_1(x)| < \frac{\beta}{2}$ and we may take $\epsilon_1 = \frac{\beta}{2}$. Observe that for $|x - x_0| \leq \alpha$ we have

$$|u_1(x) - y_0| < |u_1(x) - u(x)| + |u(x) - y_0|$$

$$< \frac{\beta}{2} + \frac{\beta}{2} < \beta$$

so that $u_1(x)$ and $u(x)$ are both in the β neighborhood of y_0. Consequently, we may still use the upper bound Q for $|g'(\eta_2)|$ in

$$e_2(x) = \int_{x_0}^{x} f(\xi)g'(\eta_2)[u_1(\xi) - u(\xi)]d\xi$$

to obtain

$$|e_2(x)| < \alpha M\epsilon_1 < \frac{\beta}{4} .$$

Again, we have

$$|u_2(x) - y_0| \leq |u_2(x) - u(x)| + |u(x) - y_0|$$

$$< \frac{\beta}{4} + \frac{\beta}{2} < \beta .$$

For the k-th iterant we obtain, inductively,

$$|e_k(x)| < \frac{\beta}{2^k} ,$$

and

$$|u_k(x) - y_0| \le |u_k(x) - u(x)| + |u(x) - y_0|$$

$$< \frac{\beta}{2^k} + \frac{\beta}{2} < \beta , \qquad (k \ge 1) .$$

Thus the conditions for obtaining the necessary bounds are satisfied for each iteration.

8. Consider the differential equation $y' = 4x\sqrt{y}$ which has more than one solution $u : x \to y$ satisfying the initial condition $u(x) = 0$ at $x = 0$, for example, $u : x \to x^4$ and $u : x \to 0$. In view of Theorem 13-2, how can this be possible?

The conditions of the theorem are not satisfied. With $f : x \to 2x$, $g : y \to 2\sqrt{y}$, observe that g' is not bounded in any neighborhood of $x = 0$. Consequently, there is no reason to expect the solution to be unique. Compare TC10-9, p. 778.

TC13-3. Taylor's Theorem.

The text uses the simplest possible approach, for us, to the Taylor approximation theorem. The estimate of the Taylor remainder is the one most commonly used in practical problems and is easily obtained by the device of repeated integration which the student has already seen in Chapter 8. Among the important theorems covered in the exercises, we have given for proof the exact forms of the remainder, the Cauchy, Lagrange, and integral remainders of Numbers 8 and 9. The integral remainder is the only explicit one, since the others involve arguments of f known only to exist by the Law of the Mean or the Mean Value Theorem. These exact forms are quite useful in theoretical investigations, the Lagrange form being the one most often encountered. In establishing error estimates for the integration formulas of Section 13-4 (Solution to Exercises 13-4, No. 5) we have used the integral form of the remainder. In Number 9(c) the Cauchy remainder was used to establish the convergence of the power series for $\arcsin x$, when $|x| < 1$. It is a conventional text problem to use the Cauchy remainder to prove convergence for the general binomial expansion of $(1 + x)^{\alpha}$ to $(1 + x)^{\alpha}$ when $|x| < 1$ and this would seem to be its most significant use. However, the rapidity of convergence and radius of convergence (but not convergence to the function) for the binomial series are most easily found by the Ratio Test of Chapter 14, so it does not seem worthwhile to stress the Cauchy remainder. (See Exercises 14-M, No. 13.)

Solutions Exercises 13-3

1. How many terms of the Taylor expansion of $\sqrt{1 - x}$ in the neighborhood of $x = 0$ should be used to give $\sqrt{7} = \frac{8}{3}\sqrt{1 - \frac{1}{64}}$ accurately to five decimal places?

For $f : x \to \sqrt{1 - x}$ obtain

$$f'(x) = -\frac{1}{\sqrt{1 - x}}$$

$$f''(x) = -\frac{1}{2(1 - x)^{3/2}}$$

$$f'''(x) = -\frac{1 \cdot 3}{2^2(1 - x)^{5/2}}$$

$$- - -$$

$$f^{(k)}(x) = -\frac{1 \cdot 3 \cdot 5 \cdots (2k - 3)}{2^{k-1}(1 - x)^{(2k-1)/2}} .$$

Since $f^{(k)}(x)$ is a negative decreasing function we have for $0 \leq x \leq \frac{1}{64}$

$$|f^{(k)}(x)| \leq \frac{1 \cdot 3 \cdot 5 \cdots (2k-3)}{2^{k-1}(1-\frac{1}{64})^{(2k-1)/2}} .$$

Consequently, the error in taking $f_n(\frac{1}{64})$ for $f(\frac{1}{64})$ satisfies $(k = n + 1)$

$$\text{(i)} \qquad |R_n(\tfrac{1}{64})| < \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n}(\frac{64}{63})^{2n+1} \frac{1}{(64)^{n+1}} .$$

Ignore the coefficient of $\frac{1}{(64)^{n+1}}$ for a first estimate. Since we want an error less than $\frac{1}{2 \times 10^5}$ choose n so that

$$(64)^{n+1} = 2^{6n+6} > 2^{20} > 10^6 > 2 \times 10^5$$

where we use $2^{10} = 1024 > 10^3$. Thus, try $n = 3$. For $n = 3$, obtain for the upper estimate in (i)

$$\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}(\frac{64}{63})^7 \frac{1}{(64)^4} < \frac{5}{16}(1.2) \frac{1}{2^{24}} < \frac{1}{2 \times 10^7}$$

which yields seven decimal place accuracy. In this, no allowance is made for round-off error.

2. Obtain $\sqrt[3]{9}$ to three decimal place accuracy.

Use $9 = 8 + 1$ to obtain

$$\sqrt[3]{9} = 2 \sqrt[3]{1 + \frac{1}{8}}$$

and expand $2(1 + x)^{1/3}$ in the neighborhood of $x = 0$ to enough terms to obtain the desired accuracy. For $f : x \rightarrow (1 + x)^{1/3}$,

$$f'(x) = \frac{2}{3(1+x)^{2/3}}$$

$$f''(x) = -\frac{4}{3^2(1+x)^{5/3}}$$

$$f'''(x) = \frac{20}{3^3(1+x)^{5/3}}$$

f''' is decreasing on $[0, \frac{1}{8}]$. Therefore take

$$M_3 = f'''(0) = \frac{20}{27} .$$

From (4),

$$\begin{aligned} |R_2(\tfrac{1}{8})| &\le \frac{20}{27} \cdot \frac{1}{6} \cdot \frac{1}{8^3} \\ &\le \frac{20}{162} \cdot \frac{1}{8^3} \\ &< \frac{20}{160} \cdot \frac{1}{8^3} \\ &\le \frac{1}{2^2} \cdot \frac{1}{2^{10}} \\ &< \frac{1}{4 \times 10^3} . \end{aligned}$$

Thus the desired accuracy is obtained for $n = 2$;

$$\begin{aligned} \sqrt[3]{9} &\approx 2(1 + \frac{1}{24} - \frac{1}{576}) \\ &\approx 2.080 \end{aligned}$$

3. Give $\tan \frac{1}{100}$ accurately to 5 decimal places.

For $f : x \rightarrow \tan x$ obtain the successive derivatives

$$\begin{aligned} f'(x) &= \frac{1}{\cos^2 x} \\ f''(x) &= \frac{2 \sin x}{\cos^3 x} \\ f'''(x) &= \frac{2 \cos^4 x - 6 \sin^2 x \cos^2 x}{\cos^4 x} \\ &= 8 - \frac{6}{\cos^2 x} . \end{aligned}$$

Consequently,

$$\tan x = x + R_2(x)$$

where, for $\cos^2 x > \frac{3}{5}$, $|f'''(x)| < 2$

$$|R_2(x)| < \frac{2x^3}{6} .$$

For $x = \frac{1}{100}$, the error in taking $\tan x = x$ is at most $\frac{10^{-6}}{3}$. Thus, to the desired accuracy

$$\tan\left(\frac{1}{100}\right) = .01000 .$$

4. Give the third order Taylor polynomial at $x = a$ in each following case. Obtain a formula for the general term if you perceive the pattern.

(a) $\sqrt{1 + x}$, $a = 0$

$$y = (1 + x)^{1/2}, \quad y' = \frac{1}{2}(1 + x)^{-1/2}, \quad y'' = -\frac{1}{2}\cdot\frac{1}{2}(1 + x)^{-3/2},$$

$$y''' = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}(1 + x)^{-5/2}, \ \ldots,$$

$$y^{(k)} = \frac{(-1)^{k+1}[1\cdot 3\cdot 5 \cdots (2k - 3)]}{2^k}(1 + x)^{-(2k-1)/2}, \ \ldots, \text{ where}$$

$k > 0$. Consequently,

$$y = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \cdots$$

$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}[1\cdot 3\cdot 5 \cdots (2k - 3)]}{2^k}\frac{x^k}{k!}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(2k - 1)!}{k\, 2^{2k-1}(k!)^2} x^k .$$

(b) $\sqrt{1 + x^2}$, $a = 0$

Replace x by x^2 in the solution of Part (a) to obtain

$$y = 1 + \frac{x^2}{2} - \frac{x^4}{4} + \frac{3x^6}{8} - \cdots .$$

To justify this procedure see Number 7.

(c) $\tan x$, $a = 0$

Take the derivatives from the solution of Number 3 to obtain

$$\tan x = x + \frac{x^3}{3} + \cdots .$$

(d) $\dfrac{1}{\cos x}$, $a = 0$

$$y = \frac{1}{\cos x} , \ y' = \frac{\sin x}{\cos^2 x} , \ y'' = \frac{2}{\cos^3 x} - \frac{1}{\cos x} , \ y''' = \frac{6 \sin x}{\cos^4 x} - \frac{\sin x}{\cos^2 x} .$$

Consequently,

$$\frac{1}{\cos x} = 1 + \frac{x^2}{x} + \dots .$$

(e) $\dfrac{1}{\sin x}$, $a = \dfrac{\pi}{2}$

Observe that $\sin x = \cos(\frac{\pi}{2} - x)$ and use the solution of Part (d) to obtain

$$\frac{1}{\sin x} = \frac{1}{\cos(\frac{\pi}{2} - x)} = 1 + \frac{1}{2}(x - \frac{\pi}{2})^2 + \dots .$$

For justification, see Number 7.

(f) $\log x$, $a = 1$

$$y = \log x , \ y' = \frac{1}{x} , \ y'' = -\frac{1}{x^2} , \ y''' = \frac{2}{x^3} , \ \dots ,$$

$$y^{(k)} = \frac{(-1)^{k-1}(k-1)!}{x^{k-1}} , \ \dots ,$$

where $k > 0$. Consequently,

$$\log x = (x - 1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \dots + \frac{(-1)^{k-1} x^k}{k} + \dots .$$

Compare this result with that of Exercises 8-2, Number 6(a) .

(g) $\dfrac{1}{\sqrt{1 + x}}$, $a = 0$

Take the derivatives from the solution of Part (a) to obtain for

$$y = \frac{1}{\sqrt{1 + x}} = 2D(\sqrt{1 + x}),$$

$$y^{(k)} = \frac{(-1)^k[1\cdot 3\cdot 5 \cdots (2k-1)](1+x)^{(2k+1)/2}}{2^{k+1}}$$

$$y = 1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5}{16}x^3 + \dots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k[1\cdot 3\cdot 5 \cdots (2k-1)]}{2^{k+1}} .$$

(h) sinh log x , a = 1

Observe that for $x > 0$

$$\sinh \log x = \frac{1}{2}(x - \frac{1}{x}) .$$

From $D^n(\frac{1}{x}) = \frac{(-1)^n(n!)}{x^{n+1}}$ obtain

$$\sinh \log x = (x-1) + \frac{(x-1)^2}{2} - \frac{(x-1)}{2} + \dots + \frac{(-1)^k(x-1)^k}{2} + \dots .$$

(i) $4x^3 + 7x^2 + 2x + 5$, $a = -1$

Note that $R_4(x) = 0$ for all x . Consequently the third order Taylor polynomial is an exact representation

$$4x^3 + 7x^2 + 2x + 5 = 6 - 5(x+1)^2 + 4(x+1)^3 .$$

(j) log cos x , a = 0

Observe that $D \log \cos x = -\tan x$ and use (c) to obtain

$$\log \cos x = -\frac{x^2}{2} + \dots .$$

5. (a) Complete the proof of Theorem 13-3 by induction for $b > a$. The case $b = a$ is trivial.

For $n = 0$, the theorem states

$$f(b) = f(a) + R_0(b) ,$$

where

$$|R_0(b)| \le M_1(b - a) .$$

This result has already been proved in the text as (6) when $n = 0$. As a consequence, for any $t \in [a,b]$

(i) $$f(t) = f(a) + R_0(t) ,$$

where

(ii) $$|R_0(t)| \le M_1(t - a) .$$

since the same bound M_1 holds for $|f(x)|$ on $[a,t]$ as on the larger interval $[a,b]$. Now suppose the theorem holds for the k-th Taylor polynomial $(k < n)$:

(iii) $$|f(t) - f_k(t)| = \left|f(t) - \sum_{j=0}^{k} \frac{f^{(j)}(a)}{j!}(t - a)^j\right| \le \frac{M_{k+1}}{(k+1)!}|t - a|^{k+1},$$

where note in this case that $|t - a|^{k+1} = (t - a)^{k+1}$. Since $k < n$, the function f' also has $k + 1$ continuous derivatives, and f may be replaced by f' to yield

$$|f'(t) - f_k'(t)| = \left|f'(t) - \sum_{i=1}^{k+1} \frac{f^{(i)}(a)}{(i - 1)!}(t - a)^{i-1}\right|$$

$$\le \frac{M_{k+2}}{(k + 1)!}(t - a)^{k+1} .$$

Consequently, for $x \in [a,b]$ (compare Exercises 6-5, No. 6)

$$\left|\int_a^x [f'(t) - f_k'(t)]dt\right| \le \int_a^x |f'(t) - f_k'(t)|dt$$

$$\le \int_a^x \frac{M_{k+2}}{(k + 1)!}(t - a)^{k+1}dt$$

$$\le \frac{M_{k+2}}{(k + 2)!}(x - a)^{k+2} .$$

Since

$$\int_a^x [f'(t) - f_k'(t)]dt = [f(x) - f(a)] - \sum_{i=1}^{k+1} \frac{f^{(i)}(a)}{i!}(x - a)^i$$

$$= f(x) - \sum_{i=0}^{k+1} \frac{f^{(i)}(a)}{i!}(x - a)^i$$

$$= f(x) - f_{k+1}(x) .$$

Thus (iii) holds when k is replaced by $k + 1$. We conclude that (iii) holds for $k = n$. Since we may take $t = b$ in (iii) the theorem follows for the case $a < b$.

(b) Give the proof for the case $b < a$.

If $a > b$, integrate from t to a for $t \in [b,a]$ in

$$-M_1 \leq f'(t) \leq M_1$$

to obtain

$$-M_1(a - t) \leq f(a) - f(t) \leq M_1(a - t) ;$$

whence,

$$|f(t) - f(a)| \leq M_1|t - a| .$$

Thus (iii) holds for $k = 0$. Now suppose (iii) holds for some k with $k < n$. Observe in this case that $|t - a|^{k+1} = (a - t)^{k+1}$. Proceed as in Part (a), now with $x \in [b,a]$, and integrate from x to a to find

$$|f(x) - f_{k+1}(x)| \leq \frac{M_{k+2}}{(k + 1)!}(a - x)^{k+2} \leq \frac{M_{k+2}}{(k + 2)!}|x - a|^{k+2} .$$

which completes the inductive argument for (iii). It follows that

$$-\frac{M_{n+1}(a - x)^{n+1}}{(n + 1)!} \leq f_n(x) - f(x) \leq \frac{M_{n+1}(a - x)^{n+1}}{(n + 1)!}$$

which immediately yields the result to be proved.

6. Show if $f^{(n+1)}(x)$ has constant sign in the interval I of Theorem 13-3 then the remainder $R_n(x)$ has the same sign as

$$f^{n+1}(x)(b - a)^{n+1} .$$

13-3

First suppose $a < b$ and $0 \leq f^{(n+1)}(x) \leq M_{n+1}$ on $[a,b]$. Then, for $t \in [a,b]$,

$$0 \leq \int_a^t f^{(n+1)}(x)dx \leq M_{n+1}(t - a) \ ;$$

whence

(i) $$0 \leq f^{(n)}(t) - f^{(n)}(a) \leq M_{n+1}(t - a) \ .$$

Thus, the first order remainder for $f^{(n)}$ is positive. Repeated integrations yield

$$0 \leq f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(t - a)^k \leq \frac{M_{n+1}}{(n + 1)!}(t - a)^{n+1} \ .$$

If $a < b$, but $-M_{n+1} \leq f^{(n+1)}(x) \leq 0$, then (i) is replaced by

$$-M_{n+1}(t - a) \leq f^{(n)}(t) - f^{(n)}(a) \leq 0$$

and the argument goes as before.

Suppose now that $b < a$ and f^{n+1} is nonnegative, $0 \leq f^{(n+1)}(x) \leq M_{n+1}$. Then for $t \in [b,a]$

$$0 \leq \int_t^a f^{(n+1)}(x)dx \leq M_{n+1}(a - t)$$

whence,

$$0 \leq f^{(n)}(a) - f^{(n)}(t) \leq M_{n+1}(a - t)$$

or

$$M_{n+1}(t - a) \leq f^{(n)}(t) - f^{(n)}(a) \leq 0 \ .$$

Again, the result is obtained by repeated integrations. Similar arguments yield the result when M_{n+1} is nonpositive.

Alternatively, use the Lagrange form of the remainder, Number 8.

7. In Example 13-3b we found approximating polynomials for arc sin. We did not acutally prove that these are Taylor polynomials by verifying that the coefficients satisfy (1b). Show that these are Taylor polynomials by proving the following general uniqueness theorem. If f has $n + 1$ continuous derivatives and there exists a neighborhood of a where

$$f(x) = c_0 + c_1(x - a) + \dots + c_n(x - a)^n + Q_n(x)$$

where $|Q_n(x)| \le K_1|x - a|^{n+1}$, then

$$c_k = \frac{f^{(k)}(a)}{k!}, \qquad (i = 0,1,2,\dots,n) .$$

Let $f_n(x) = b_0 + b_1(x - a) + \dots + b_n(x - a)^n$ be the n-th order Taylor polynomial; i.e., let $b_k = \frac{f^{(k)}(a)}{k!}$. Then by Theorem 13-3

$$f(x) = f_n(x) + R_n(x)$$

where $|R_n(x)| \le K_2|x - a|^{n+1}$, (here $K_2 = \frac{M_{n+1}}{(n + 1)!}$). Taking the difference in the two representations for f we obtain

(i) $0 = (c_0 - b_0) + (c_1 - b_1)(x - a) + \dots + (c_n - b_n)(x - a)^n + P_n(x)$,

where $|P_n(x)| = |Q_n(x) - R_n(x)| \le (K_1 + K_2)|x - a|^{n+1}$. Thus $\lim_{x\to a} \frac{P_n(x)}{(x - a)^k} = 0$ for $k \le n$. Now, take $x = a$ in (i) to obtain $c_0 = b_0$. Next, divide in (i) by $x - a$ and take the limit as x approaches a to obtain $c_1 = b_1$. Repeat, dividing by the consecutive powers of $x - a$ to obtain $c_k = b_k$ for $k = 0, 1, 2, \dots, n$.

8. Lagrange obtained a form of the remainder $R_n(b)$ in (3) which generalizes the Law of the Mean. For this, apply the Law of the Mean to the function

(i) $$g(x) = \sum_{k=0}^{n} \frac{(b - x)^k}{k!} f^{(k)}(x) + A(b - x)^{n+1}$$

on the interval $[a,b]$ with the constant A chosen so that $g(a) = g(b) = f(b)$ and verify that $A = \frac{f^{(n+1)}(\xi)}{(n + 1)!}$ where ξ lies between a and b. Thus the Lagrange form of the remainder is $R_n(x) = \frac{f^{(n+1)}(\xi)}{(n + 1)!}(x - a)^{n+1}$. What conditions must f and its derivatives satisfy for this result?

Differentiate in (i) to obtain

$$g'(x) = \sum_{k=0}^{n} \frac{(b - x)^k}{k!} f^{(k+1)}(x)$$

$$- \sum_{k=1}^{n} \frac{(b - x)^{k-1}}{(k - 1)!} f^{(k)}(x)$$

$$- A(n + 1)(b - x)^n$$

or

$$\text{(ii)} \qquad g'(x) = \frac{(b - x)^n}{n!} f^{(k+1)}(x) - A(n + 1)(b - x)^n .$$

By the Law of the Mean there must exist some number ξ between a and b such that

$$g'(\xi) = \frac{g(b) - g(a)}{b - a} = 0 .$$

Enter ξ in (ii) to obtain

$$A = \frac{f^{(k+1)}(\xi)}{(n + 1)!} .$$

In the course of the proof we have assumed only that f is continuous on $[a,b]$ and has derivatives of orders up to $n + 1$ on (a,b).

9. (a) Show that the remainder in Taylor's Theorem can be written in the form

$$R_n(b) = \frac{1}{n!} \int_a^b (b - x)^n f^{n+1}(x)dx ,$$

where it is assumed that f has $n + 1$ continuous derivatives. (Hint: use induction and integration by parts. Compare Chapter 10, Miscellaneous Exercises, No. 20.)

Observe first that

$$f(b) = f(a) + R_0(b) = f(a) + \int_a^b f'(x)dx .$$

Integrate by parts using $u = f'(x)$, $v = x - b$ to obtain

$$R_0(b) = f'(x)(x - b)\Big|_a^b - \int_a^b (x - b)f''(x)dx$$

$$= f'(a)(b - a) + \int_a^b (b - x)f''(x)dx ;$$

whence,

$$f(b) = f(a) + f'(a)(b - a) + \int_a^b (b - x)f''(x)dx .$$

Now, suppose

$$\text{(i)} \qquad f(b) = \sum_{j=0}^{k} \frac{f^{(j)}(a)(b - a)^j}{j!} + R_k(b) .$$

Integrate by parts using $u = f^{(k+1)}(x)$,

$$dv = (-1)^k \frac{(x - b)^k}{k!} dx , \quad v = (-1)^k \frac{(x - b)^{k+1}}{(k + 1)!}$$

to obtain

$$R_k(b) = (-1)^k f^{(k+1)}(x) \left. \frac{(x - b)^{k+1}}{(k + 1)!} \right|_a^b - \frac{(-1)^k}{(k + 1)!} \int_a^b (x - b)^{k+1} f^{(k+2)}(x)dx$$

$$= f^{(k+1)}(a) \frac{(b - a)^{k+1}}{(k + 1)!} + R_{k+1}(b) .$$

Consequently (i) holds with k replaced by $k + 1$.

(b) From the integral form of the remainder obtain Cauchy's remainder

$$R_n(b) = \frac{1}{n!}(b - a)(b - u)^n f^{(n+1)}(u)$$

where u lies between a and b .

Use the Mean Value Theorem of Integral Calculus (Exercises 6-4, No. 20).

(c) Use Cauchy's form of the remainder to prove the claim of the text for the Taylor expansion of arcsin in Example 13-3, that $\lim_{n\to\infty} R_n(x) = 0$ for $|x| < 1$.

Use Cauchy's remainder for

$$g(t) = \frac{1}{\sqrt{1 + t}} .$$

Since, for $0 < t < 1$

$$g^{(n+1)}(t) = \frac{1 \cdot 3 \cdot 5 \cdots (2n + 1)}{2^{n+1}(1 - t)^{(2n+3)/2}}$$

the Cauchy remainder for g_n is

$$Q_n(t) = \frac{1\cdot 3\cdot 5 \cdots (2n+1)}{2^{n+1}(1-u)^{(2n+3)/2}} \frac{t(t-u)^n}{(n!)}$$

where u lies in $(0,t)$. Consequently,

$$|Q_n(t)| = \left|\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdot\ldots\frac{2n+1}{2n}\frac{t(t-u)^n}{(1-u)^{(2n+3)/2}}\right|$$

$$= \left|\frac{t}{2}\right|\left|(1+\frac{1}{2})t\right| \ldots \left|(1+\frac{1}{2n})t\right| \frac{(1-\frac{u}{t})^n}{(1-u)^{(2n+3)/2}}$$

since $0 < \frac{u}{t} < 1$. Now, since $0 < t < 1$, $\frac{u}{t} > u$ and $1 - \frac{u}{t} < 1 - u$ and

$$|Q_n(t)| < \left|\frac{t}{2}\right|\left|(1+\frac{1}{2})t\right| \ldots\left|(1+\frac{1}{2n})t\right|(1-u)^{-3/2} .$$

Now suppose $t \leq s < 1$. Set $r = \frac{1+|s|}{2}$. Observe that $s < r < 1$. For n sufficiently large, $n > \nu = \left[\frac{|s|}{1-|s|}\right]$,

$$\left|(1+\frac{1}{2k})t\right| < r .$$

Consequently,

$$|Q_n(t)| < \frac{1}{2}(1+\frac{1}{2})(1+\frac{1}{2N})r^{n-\nu}(1-r)^{-3/2}$$

$$< Cr^n$$

where C is a constant independent of n. Now $R_n(x)$ for the arcsin series is given by Equation (11) as

$$R_n(x) = \int_0^x Q_n(z^2)dz$$

(compare No. 7); hence

$$|R_n(x)| < C|x|r^n ,$$

where $|x| < 1$, $r = \frac{1+x^2}{2}$. Since $r < 1$ it follows from the Squeeze Theorem that

$$\lim_{n\to\infty} R_n(x) = 0 .$$

10. The eccentricity e of an ellipse is given by $e^2 = 1 - \frac{b^2}{a^2}$ where a is the semi-major axis and b, the semi-minor axis. The circle, $a = b$, has eccentricity $e = 0$, thus e measures the departure from circular symmetry. Obtain the arclength of the ellipse in the form $s = a\,f(e)$ and expand f in powers of e to sixth order. (As we mentioned in Section 12-4(iii), the integral for the arclength of an ellipse cannot be written in terms of elementary functions. The solution of this exercise yields precise estimates of the arclength provided the eccentricity is not too large.)

Use the parametric equations

$$x = a \cos \phi , \; y = b \sin \phi$$

for the ellipse. Then

$$\begin{aligned}
\text{(i)} \qquad s &= 4 \int_0^{\pi/2} \sqrt{a^2 \sin^2 \phi + b^2 \cos^2 \phi}\; d\phi \\
&= 4 \int_0^{\pi/2} \sqrt{a^2 + (b^2 - a^2)\cos^2 \phi}\; d\phi \\
&4a \int_0^{\pi/2} \sqrt{1 - e^2 \cos^2 \phi}\; d\phi .
\end{aligned}$$

Now, from

$$\sqrt{1 - t} = 1 - \frac{t}{2} - \frac{t^2}{8} - \frac{t^3}{16} - \cdots$$

obtain

$$\sqrt{1 - e^2 \cos^2 \phi} = 1 - \frac{e^2 \cos^2 \phi}{2} - \frac{e^4 \cos^4 \phi}{8} - \frac{e^6 \cos^6 \phi}{16} - \cdots .$$

From Equation (5a) of Section 10-6 (page 574), we have on integrating,

$$s = 2\pi a\left[1 - \frac{e^2}{4} - \frac{3}{64} e^4 - \frac{5}{256} e^6 - \ldots\right] .$$

---

11. A function f is said to have a zero of order k at $x = a$ if $0 = f(a) = f'(a) = f''(a) = \ldots = f^{(k-1)}(a)$ and $f^{(k)}(a) \neq 0$; the leading term in the Taylor expansion of f at a is then $\frac{f^{(k)}(a)}{n!}(x - a)^k$. Prove if f has a first order zero at $x = a$ then the function g given by $g(x) = [f(x)]^n$ has a zero of order n . (Hint: Use the Lagrange remainder of No. 8 for f .)

If f has a first order zero then

$$f(x) = f'(a)(x - a) + R_1(x)$$

where $R_1(x) = \frac{f''(\xi)}{2}(x - a)^2$ for some ξ between a and x. Consequently, we have the existence of the limit,

$$\lim_{x \to a} \frac{R_1(x)}{(x - a)^2} = \frac{1}{2} f''(a) .$$

It follows that the function Q_1 defined by

$$Q(x) = \begin{cases} \frac{R_1(x)}{x - a} , & \text{for } x \neq a \\ 0 , & \text{for } x = a \end{cases}$$

is differentiable. Thus

$$\begin{aligned} f(x) &= [f'(a) + Q(x)](x - a) \\ &= (x - a)\phi(x) \end{aligned}$$

where $\phi(a) \neq 0$. Consequently,

$$\begin{aligned} g(x) = [f(x)]^n &= (x - a)^n[\phi(x)]^n \\ &= (x - a)^n\{[\phi(a)]^n + n[\phi(u)]^{n-1}\phi'(u)(x - a)\} \end{aligned}$$

where u lies between x and a . Now apply Number 7 (thus it is assumed that g has $n + 1$ continuous derivatives) to obtain the desired result.

12. Two curves are said to have a contact of order n at a point X_0 if n is the largest integer for which the curves have parametrizations $\vec{X} = \vec{r}(t)$, $\vec{Y} = \vec{q}(t)$, respectively, with $\vec{X}_0 = \vec{r}(t_0) = \vec{q}(t_0)$ such that $\vec{r}'(t_0) \neq \vec{0}$ and $\vec{q}'(t_0) \neq \vec{0}$ and $\vec{r}^{(k)}(t_0) = \vec{q}^{(k)}(t_0)$ for $k = 0 , 1 , \ldots , n$ and $\vec{r}^{(n+1)}(t_0) \neq \vec{q}^{(n+1)}(t_0)$. Taylor's Theorem can easily be extended component-by-component to vector functions so that this condition may also be given in terms of Taylor polynomials as before.

(a) Prove that if t is replaced by an equivalent parameter, the order of contact is unaffected.

Set $t = \phi(u)$ where $t_0 = \phi(u_0)$ and $\phi'(u_0) > 0$. Introduce the vector function $\vec{p} = \vec{r} - \vec{q}$. We are given that $\vec{p}^{(k)}(t_0) = \vec{0}$ for $k = 0 , 1 , 2 , \ldots , n$ and $\vec{p}^{(k+1)}(t_0) \neq \vec{0}$. For the composite function $\vec{\pi} : u \to \vec{p}(\phi(u))$ we wish to prove similarly that $\vec{\pi}^{(k)}(u_0) = \vec{0}$ for $k = 0, 1, 2, \dots, n$ and $\vec{\pi}^{(k+1)}(u_0) \neq \vec{0}$. (In this we assume that ϕ has as many continuous derivatives as we may need.) Note that

$$\begin{aligned}
\vec{\pi}(u) &= \vec{p}(t) \\
\vec{\pi}'(u) &= \vec{p}'(t)\phi'(u) \\
\vec{\pi}''(u) &= \vec{p}''(t)[\phi(u)]^2 + \vec{p}'(t)\phi''(u)
\end{aligned}$$

etc. We prove in general, that the k-th derivative of $\vec{\pi}$, where $k \geq 1$, has the form

$$\text{(i)} \quad \vec{\pi}^{(k)}(u) = \vec{p}^{(k)}(t)[\phi'(u)]^k + \vec{p}^{(k-1)}(t)f_1(u) + \vec{p}^{(k-2)}(t)f_2(u) + \dots + \vec{p}'(t)f_{k-1}(u) .$$

The result has been proved for $k = 1$. The argument is inductive. If (i) holds, then on differentiation with respect to u, we obtain

$$\begin{aligned}
\text{(ii)} \quad \vec{\pi}^{(k+1)}(u) = \vec{p}^{(k+1)}(t)[\phi'(u)]^{k+1} \\
+ \vec{p}^{(k)}(t)\{k[\phi'(u)]^{k-1}\phi''(u) + f_1(u)\phi'(u)\} \\
+ \vec{p}^{(k-1)}(t)\{f_1'(u) + f_2(u)\phi'(u)\} \\
+ \dots + \vec{p}'(t)f_{k-1}'(u) ,
\end{aligned}$$

which has the same general form as (i) with k replaced by $k + 1$. We have $\vec{\pi}(u_0) = \vec{p}(t_0) = \vec{0}$. On taking $u = u_0$, $t = t_0$ in (i) we obtain further $\vec{\pi}^{(k)}(u_0) = 0$, for $k = 1, 2, \dots, n$, and

$$\vec{\pi}^{(n+1)}(u_0) = \vec{p}^{n+1}(t_0)[\phi'(u_0)]^{n+1} \neq \vec{0} .$$

(b) Let s and σ be arclength along the curves $\vec{X} = \vec{r}(t)$ and $\vec{Y} = \vec{q}(t)$. Show if the curves have contact of order n as defined in Part (a) then the parameter of Part (a) may be replaced by arclength; i.e., for

$$s = \int_{t_0}^{t} |r'(\tau)|d\tau \quad \text{and} \quad \sigma = \int_{t_0}^{t} |q'(\tau)|d\tau ,$$

we have $\left.\frac{d^k\vec{X}}{ds^k}\right|_{s=0} = \left.\frac{d^k\vec{Y}}{d\sigma^k}\right|_{\sigma=0}$ where $k = 0, 1, 2, \ldots, n$, and

$$\left.\frac{d^{n+1}\vec{X}}{ds^{n+1}}\right|_{s=0} \neq \left.\frac{d^{n+1}\vec{Y}}{d\sigma^{n+1}}\right|_{\sigma=0}$$

This result implies that order of contact has an invariant meaning, independent of the choice of the coordinate frame or the method of parametrization. Employ the chain rule to obtain

$$\frac{d\vec{X}}{ds} = \frac{d\vec{X}}{dt}\Big/\frac{ds}{dt} = \frac{\vec{X}'}{|\vec{X}'|}$$

where the prime indicates differentiation with respect to t. Differentiate repeatedly with respect to s to obtain

$$\frac{d^2\vec{X}}{ds^2} = \frac{\vec{X}''}{|\vec{X}'|^2} - \frac{\vec{X}'(\vec{X}'' \cdot \vec{X}')}{|\vec{X}'|^3},$$

(compare Exercises 11-5, No. 4(b)) and, in general,

$$\frac{d^k\vec{X}}{ds^k} = R_k(|\vec{X}'|, \vec{X}', \vec{X}'', \ldots, \vec{X}^{(k)})$$

where R_k is a rational combination of its arguments. Similarly,

$$\frac{d^k\vec{Y}}{d\sigma^k} = R_k(|\vec{Y}'|, \vec{Y}', \vec{Y}'', \ldots, \vec{Y}^{(k)}) .$$

Since the first n derivatives of $\vec{X}$ and $\vec{Y}$ are the same at $t = 0$, it follows that $\frac{d^k\vec{X}}{ds^k} = \frac{d^k\vec{Y}}{d\sigma^k}$ for $k = 0, 1, 2, \ldots, n$. Since n is the largest such integer, we conclude that $\frac{d^{n+1}\vec{X}}{ds^k} \neq \frac{d^{n+1}\vec{Y}}{d\sigma^{n+1}}$. It follows that order of contact could have been defined in Part (a) as the order to which the Taylor polynomials coincide when arclength is taken as the parameter.

(c) Show that the curves $y = f(x)$ and $y = g(x)$ have a contact of order n at $x = a$ if, and only if, $f - g$ has a zero of order $n + 1$ at a.

First from the definition given in Part (a) observe, with x taken in place of t as parameter, that if $f - g$ has a zero of order $n + 1$ then the order of contact is at least n. We shall prove, using Part (b), that if the contact is of order n, then $f^{(k)}(x_0) = g^{(k)}(x_0)$ for $k = 0, 1, 2, \ldots, n$, namely that $f - g$ has a zero of order at least $n + 1$. From this pair of propositions the result follows: if $f - g$ has a zero of order $n + 1$, then the contact is of order n or higher, but the order of contact cannot be higher than n for then the order of the zero would be higher than $n + 1$; the converse follows similarly.

For the proof set $y = f(x)$, $v = g(u)$ for clarity and take s and σ, respectively, as arclength parameters on the curves with $s = \sigma = 0$ at the point of contact. If the contact is of order n, then by the result of Part (b) with $\vec{X} = (x,y)$, $\vec{Y} = (u,v)$ we have

$$\left.\frac{d^k\vec{X}}{ds^k}\right|_{s=0} = \left.\frac{d^k\vec{Y}}{d\sigma^k}\right|_{\sigma=0}, \text{ for } k = 0, 1, 2, \ldots, n,$$

while $\dfrac{d^{n+1}\vec{X}}{ds^{n+1}} \neq \dfrac{d^{n+1}\vec{Y}}{d\sigma^{n+1}}$. Now let $x^{(k)}$, $y^{(k)}$ denote k-th derivatives with respect to s and $u^{(k)}$, $v^{(k)}$, k-th derivatives with respect to σ. We have $x' = \dfrac{1}{\sqrt{1 + (y')^2}} \neq 0$ and, similarly, $u' \neq 0$.

Now, $\dfrac{dy}{dx} = \dfrac{y'}{x'}$, $\dfrac{d^2y}{dx^2} = \dfrac{x'y'' - y'x''}{(x')^3}$, ..., and, in general

$$\frac{d^ky}{dx^k} = R_k(x',y',x'',y'',\ldots,x^{(k)},y^{(k)})$$

where R_k is a rational combination of its arguments. Similarly

$$\frac{d^kv}{du^k} = R_k(u',v',u'',v'',\ldots,u^{(k)},v^{(k)}) .$$

Now, when $\sigma = s = 0$, we have $u^{(k)} = x^{(k)}$ and $v^{(k)} = y^{(k)}$, for $k = 0, 1, 2, \ldots, n$. It follows that $\left.\dfrac{d^ky}{dx^k}\right|_{x=x_0} = \left.\dfrac{d^kv}{du^k}\right|_{\xi=x_0}$, for $k = 0, 1, 2, \ldots, n$; namely that $f - g$ has a zero of order $n + 1$ or higher, as we sought to prove.

13. Infinite order of contact at a given point does not necessarily imply that the two curves coincide on any neighborhood of the point. Show that the curve

$$y = \begin{cases} e^{-1/x^2} & , \text{ for } x \neq 0 \\ 0 & , \text{ for } x = 0 \end{cases}$$

has a contact of infinite order with the x-axis at $x = 0$.

Show that all the derivatives of y vanish at $x = 0$. Thus, with $z = \frac{1}{x^2}$

$$y_0' = \lim_{x \to 0} \frac{e^{-1/x^2}}{x} = \lim_{z \to \infty} \frac{\sqrt{z}}{e^z} = 0$$

from Lemma 8-3. (Here $y_0^{(k)} = \left.\frac{d^ky}{dx^k}\right|_{x=0}$.) Thus

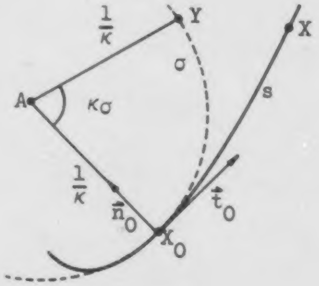
$$y' = \begin{cases} \frac{2}{x^3} e^{-1/x^2} & , \text{ for } x \neq 0 \\ 0 & , \text{ for } x = 0 \end{cases}$$

is continuous. In general, the derivatives of y have the form

$$\text{(i)} \qquad y^{(k)} = \begin{cases} \left(\frac{a_1}{x^{2k+1}} + \frac{a_2}{x^{2k}} + \dots + \frac{a_k}{x^{k+1}}\right) e^{-/x^2} & , \text{ for } x \neq 0 \\ 0 & , \text{ for } x = 0 \end{cases}$$

and by differentiation and application of Lemma 8-3 as above it follows inductively that if (i) is satisfied for any value of k , it is satisfied for all larger values.

14. (a) Show that the osculating circle to a curve at a given point has a contact of order 2 or more.



Let s denote arclength measured along the curve $\vec{X} = \vec{r}(s)$, and σ , arclength along the osculating circle, $\vec{Y} = \vec{q}(\sigma)$, to the curve $\vec{X} = \vec{r}(s)$ at $s = 0$. The central angle subtended by the arc $\widehat{X_0Y}$ is $\kappa\sigma$ (see figure) where κ is the curvature, that is the reciprocal of the radius. To second order, we have for a point X of the curve,

$$\text{(i)} \qquad \vec{X} = \vec{X}_0 + s\vec{X}'_0 + \frac{s^2}{2}\vec{X}''_0 + \dots$$

or

$$\vec{X} = \vec{X}_0 + s\vec{t}_0 + \frac{s^2}{2}\kappa\,\vec{n}_0 + \dots .$$

For a point Y of the osculating circle we have on resolving into component in the directions of $\vec{t}$ and $\vec{n}$

$$\vec{Y} = \vec{A} + \frac{\sin \kappa\sigma}{\kappa}\vec{t}_0 - \frac{\cos \kappa\sigma}{\kappa}\vec{n}_0 ,$$

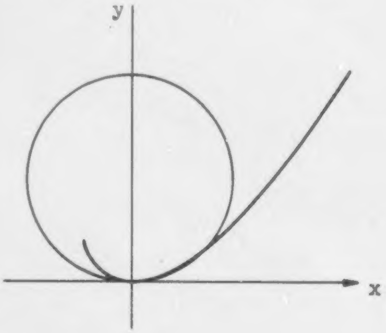
where $\vec{A} = \vec{X}_0 + \frac{\vec{n}_0}{\kappa}$ is the center of the circle. Expand $\sin \kappa\sigma$ and $\cos \kappa\sigma$ to second order and obtain

$$\vec{Y} = (\vec{X}_0 + \frac{\vec{n}_0}{\kappa}) + \sigma\, t_0 - \frac{1}{\kappa}(1 - \frac{\kappa^2\sigma^2}{2})\vec{n}_0 + \dots$$

which coincides with (i) to the stated order.

(b) Prove that if an osculating circle to a plane curve has a contact of order 2 then it crosses the curve at the point of contact (Hint: use the result of No. 11(b)).

Choose a coordinate system in which the origin is taken at the point of contact and the x-axis is oriented in the direction of the tangent (see figure). In this coordinate system the curve and the osculating circle may be described in a neighborhood of the origin as the graphs of functions $y = f(x)$ and $y = g(x)$, respectively. Expanding to third order, we have



$$f(x) - g(x) = ax^3 + R_3(x)$$

where, by Number 11(b), $a \neq 0$. From Taylor's Theorem we have $|R_3(x)| < bx^4$ for some positive, b . Consequently, for $x \neq 0$,

$$f(x) - g(x) = ax^3[1 + \epsilon] ,$$

where

$$|\epsilon| = \left|\frac{R_3(x)}{ax^3}\right| < \frac{b|x|}{|a|} < 1$$

provided $|x| < \frac{|a|}{b}$. In the neighborhood of the point of contact defined by this inequality the sign of $1 + \epsilon$ is positive. Consequently

$$\operatorname{sgn}(f(x) - g(x)) = \operatorname{sgn} ax^3 .$$

Thus the sign of $f - g$ changes across the point of contact and we conclude that the two curves cross there.

This proof requires existence and continuity of the fourth derivatives. Alternatively, require only existence and continuity of the third derivatives and use Number 6. Since the third derivatives of $f - g$ at x_0 is not zero, continuity guarantees that it has constant sign on some neighborhood of x_0 as the proof of Number 6 requires.

Note that the argument applies to any two curves which have a contact of even order. If the contact is of odd order the two curves touch but do not cross.

15. Given $y = f(x)$ and $y = g(x)$ have contact of order n at $x = a$ and $g^{(n+1)}(a) \neq 0$, prove that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f''(x)}{g''(x)} = \dots = \frac{f^{(n+1)}(a)}{g^{(n+1)}(a)} .$$

Observe that this is not L'Hôpital's Rule in its most general form. L'Hôpital's Rule states, if $f(a) = g(a) = 0$, and $g'(x) \neq 0$ on some deleted neighborhood of a , and $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$. (The proof of L'Hôpital's Rule follows directly from the Generalized Law of the Mean, Solution to Exercises 11-M, No. 10; namely, from $\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(u)}{g'(u)}$ where u is some number between x and a .) In order to use the approach of Taylor's Theorem to obtain the first order result, we must assume the existence of a second derivative and obtain only

(i) $$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

which has meaning only if $g'(a) \neq 0$. Thus $\lim\limits_{x\sim a} \frac{f'(x)}{g'(x)}$ may exist and L'Hôpital's Rule still be applicable where (i) is not. The difference is not so important as may appear (except in certain singular cases* like $\lim\limits_{x\sim 0} x \log x = \lim\limits_{x\sim 0} \frac{x}{1/\log x}$); if $g'(a) = 0$ and the limit exists (thus $f'(a) = 0$ also), then it is usually possible to obtain the limit by going to higher derivatives as in this exercise.

Assume $n + 2$ continuous derivatives. Then by Taylor's Theorem, for $k < n + 1$

$$\frac{f^{(k)}(x)}{g^{(k)}(x)} = \frac{f^{(n+1)}(a)(x - a)^{n+1} + (n + 1 - k)!\phi_k(x)}{g^{(n+1)}(a)(x - a)^{n+1} + (n + 1 - k)!\psi_k(x)}$$

where in some neighborhood of a

$$|\phi_k(x)| < \frac{A_{n+2}|x - a|^{n+2}}{(n + 2 - k)!}$$

and

$$|\psi_k(x)| < \frac{B_{k+2}|x - a|^{n+2}}{(n + 2 - k)!} ,$$

where A and B are upper bounds for $|f^{(n+2)}(x)|$ and $|g^{(n+2)}(x)|$, respectively. Consequently,

$$\lim_{x\sim a} \frac{f^{(k)}(x)}{g^{(k)}(x)} = \lim_{x\sim a} \frac{f^{(n+1)}(a) + \epsilon_1(x)}{g^{(n+1)}(a) + \epsilon_2(x)}$$

where $\lim\limits_{x\sim a} \epsilon_1(x) = \lim\limits_{x\sim a} \epsilon_2(x) = 0$, from which the result is immediate.

Alternatively, use the Lagrange form of the remainder from Number 8 and assume only the continuity of the $(n + 1)$-th derivative.

---

*In this case, L'Hôpital's Rule yields the valid result $\lim\limits_{x\sim 0} x \log x = -\lim\limits_{x\sim 0} x (\log x)^2$ but it does not help in the evaluation of the limit. For that, L'Hôpital's Rule must be extended to the case $\lim\limits_{x\sim a} f(x) = \infty$. $\lim\limits_{x\sim a} g(x) = \infty$, where in this example $f(x) = \log x$ and $g(x) = \frac{1}{x}$.

16. In the light of Number 15 calculate the following limits.

(a) $\lim\limits_{x \to 1} \dfrac{x^n - 1}{x - 1} = n$.

(b) $\lim\limits_{x \to 1} \dfrac{1 - x}{\log x} = -1$.

(c) $\lim\limits_{x \to 0} \dfrac{\sin ax}{x} = a$.

(d) $\lim\limits_{t \to 0} \dfrac{\tan x - x}{x - \sin x} = 2$.

(e) $\lim\limits_{x \to 0} \dfrac{x - \sin x}{\sin^3 x} = \dfrac{1}{6}$.

(f) $\lim\limits_{x \to \pi} (x - \pi)\tan \dfrac{x}{2} = \lim\limits_{x \to \pi} \dfrac{x - \pi}{\cot \frac{x}{2}} = -1$.

(g) $\lim\limits_{x \to a} \dfrac{\sin x - \sin a}{x - a} = \cos a$. (The result also follows directly since the limit is $\dfrac{d}{dx}(\sin x)\Big|_{x=a}$.

(h) $\lim\limits_{x \to 1} \{1 - \dfrac{1}{\log x} + \dfrac{1}{x - 1}\} = \lim\limits_{x \to 1} \dfrac{x \log x - x + 1}{(x - 1)\log x} = \dfrac{1}{2}$.

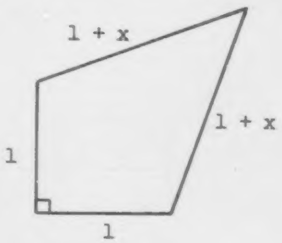
(i) $\lim\limits_{x \to 0} \dfrac{1}{x} \log \dfrac{1 - \alpha x}{1 - \beta x} = \beta - \alpha$.

17. If $f - g$ has a zero of order n at a we say that $g(x)$ approximates $f(x)$ in the neighborhood of $x = a$ with an *error of order n* . We also say $f(x) = g(x) + A(x - a)^n$ plus terms of higher order (here $A = \dfrac{g^{(n)}(a) - f^{(n)}(a)}{n!}$) .

(a) Let s_1 , s_2 , s_3 , s_4 be successive sides of a convex quadrilateral. An ancient Egyptian document gives as the formula for the area of the quadrilateral

$$\frac{1}{4}(s_1 + s_3)(s_2 + s_4) .$$

This formula is correct for rectangles but is not generally valid. For a quadrilateral with two adjacent perpendicular sides of length 1 and two other sides of length $1 + x$, (see figure). What is the order of the error of the Egyptian formula in the neighborhood of $x = 0$?



The Egyptian formula yields the approximation

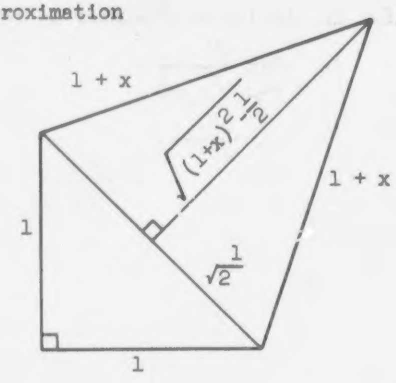
$$\alpha = \frac{(2 + x)^2}{4} = 1 + x + \frac{x^2}{4} .$$

From the adjacent figure, the correct value is found to be

$$A = \frac{1}{2} + \sqrt{\frac{(1 + x)^2}{2} - \frac{1}{4}}$$

$$= 1 + x - \frac{x^2}{2} + \dots .$$

The error is second order in x .



(b) Let s be the arclength measured from X_0 to X along a plane curve. Determine the order in s to which the arclength is approximated by the chord length $\ell = |X - X_0|$ and give the error to lowest order.

Observe from Section 11-6, Equations (10) and (11),

$$\vec{X} = \vec{X}_0 + s\vec{t}_0 + \kappa_0 \frac{s^2}{2} \vec{n}_0 + \frac{s^3}{6}(\kappa_0' \vec{n}_0 - \kappa_0^2 \vec{t}_0) + \dots$$

where $\vec{t}_0$, $\vec{n}_0$, κ_0 are, respectively, the tangent normal and curvature at $s = 0$. It follows that

$$\ell^2 = |\vec{X} - \vec{X}_0|^2 = s^2 - \frac{\kappa_0^2 s^4}{12} + \dots ,$$

whence

$$\ell = s \sqrt{1 - \frac{\kappa_0^2 s^2}{12} + \dots}$$

$$= s(1 - \frac{\kappa_0^2 s^2}{24} + \dots) .$$

The error is third order and, to lowest order, it is

$$\ell - s = - \frac{\kappa_0^2 s^3}{24} .$$

TC13-4. Numerical Integration.

In practice, it is important to be alert to other sources of error than the one we have treated in this section, the error in replacing the integrand by an interpolated polynomial. Typically, the function values y_k can only be given approximately. Round-off error, too, must be accounted for. Usually these matters are taken care of by carrying the terms of a computation to a sufficient number of decimal place.

We have not derived the rigorous estimates (6) and (14) but preferred to obtain the estimates (5) and (13) heuristically. There are good reasons for this. The heuristic approach tells us what to look for. We expand the integral and the approximation formula to the order at which the Taylor expansions first differ. The error to lowest order is then known. A rigorous error bound can then be derived by the method of Exercises 13-4, Number 5. Moreover, as we saw in Example 13-4, the nonrigorous error estimate may be more accurate than the rigorous error tolerance.

Solutions Exercises 13-4

1. (a) Estimate π by approximation to

$$\frac{\pi}{4} = \int_0^1 \frac{1}{1+x^2}\,dx \ .$$

We give the computation by Simpson's Rule for $n = 10$.

$y_0 = 1.000\ 000\ 0$ $\qquad$ $y_1 \approx .990\ 099\ 0$

$y_2 \approx .961\ 538\ 5$ $\qquad$ $y_3 \approx .917\ 431\ 2$

$y_4 \approx .862\ 069\ 0$ $\qquad$ $y_5 \approx .800\ 000\ 0$

$y_6 \approx .735\ 294\ 1$ $\qquad$ $y_7 \approx .671\ 140\ 9$

$y_8 \approx .609\ 756\ 1$ $\qquad$ $y_9 \approx .552\ 486\ 2$

$y_{10} = .500\ 000\ 0$.

Simpson's Rule gives

$$\frac{\pi}{4} \approx \frac{1}{30}[(y_0+y_{10})+4(y_1+y_3+y_5+y_7+y_9)+2(y_2+y_4+y_6+y_8)]$$

$$\approx \frac{1}{30}[1.5 + 15.7246292 + 6.3373154] = .785\ 398\ 2 \ ;$$

whence $\pi \approx 3.141593$. The result is accurate to the number of places given.

(b) Estimate π by approximation to

$$\frac{\pi}{6} = \int_0^{1/2} \frac{1}{\sqrt{1 - x^2}}\, dx .$$

This time we use Simpson's Rule for $n = 2$ and obtain $y_0 \approx 1.000$, $y_1 \approx 1.021$, $y_2 \approx 1.154$; whence

$$\frac{\pi}{6} \approx \frac{1}{12}[y_0 + 4y_1 + y_2] \approx \frac{6.238}{12}$$

and

$$\pi \approx 3.12 .$$

(c) Estimate how large n should be taken in Simpson's Rule to give π accurately to 5 places by approximation to the integral of Part (a).

For the error in π we have from (14b), $4|\epsilon| \le \frac{M_4}{45n^4}$. We estimate n by $\frac{M_4}{45n^4} \le \frac{1}{2 \times 10^5}$ or $n^4 \ge \frac{2 \times 10^5 M_4}{45} \ge \frac{4M_4}{9} \times 10^4$; whence,

(i) $$n \ge 10 \sqrt[4]{\frac{4M_4}{9}} .$$

To estimate M_4 differentiate $y = \frac{1}{1 + x^2}$ four times to obtain

$$y^{\text{iv}} = \frac{24(5x^4 - 10x^2 + 1)}{(1 + x^2)^5} = \frac{24(5u^2 - 10u + 1)}{(1 + u)^5} ,$$

where $u = x^2$. To determine the maximum of y^{iv} on $[0,1]$, locate the zeros of

$$\frac{dy^{\text{iv}}}{du} = \frac{-120(3u - 1)(u - 3)}{(1 + u)^6} .$$

The only zero in the interval occurs at $u = \frac{1}{3}$ and we have as candidates for the maximum

u	0	$\frac{1}{3}$	1
y	24	$-24(\frac{3}{4})^3$	-3

Thus the maximum occurs at $u = 0$ and we take $M_4 = 24$ in (i). Observe that $\frac{4M_4}{9} = \frac{32}{3} < 11$. So we may take

$$n \geq 10 \sqrt[4]{11} \approx 10 \times 1.82 \approx 18 \ .$$

This is actually far too large as we know from the solution of Part (a). A more realistic procedure would be to attempt to estimate ϵ with the help of (12). We would then obtain

$$\epsilon \approx \frac{h^5}{90}\{y_1^{\text{iv}} + y_3^{\text{iv}} + \dots + y_{n-1}^{\text{iv}}\} \ .$$

This suggests that the sum be approximated by a Riemann integral (see Exercises 6-4, No. 20); that is,

$$\begin{aligned} \epsilon &\approx \frac{h^4}{180} \sum_{j=1}^{n/2} f^{\text{iv}}(x_{2j-1})2h \\ &\approx \frac{h^4}{180} \int_a^b f^{\text{iv}}(x)dx \ ; \end{aligned}$$

whence

$$\text{(ii)} \qquad \epsilon \approx \frac{h^4}{180}[f'''(b) - f'''(a)] = \frac{(b-a)^4}{180n^4}[f'''(b) - f'''(a)] \ .$$

In the case under consideration

$$f'''(x) = \frac{24(x - x^3)}{(1 + x^2)^4}$$

which, for $a = 0$, $b = 1$, yields $\epsilon \approx 0$. In this case, (ii) is of no use as it stands except to indicate that (i) yields a gross overestimate of the number of partition points needed. (We could, of course, estimate the error in the error estimate which is obtained by the so-called Tangent Rule of numerical integration. This yields the error estimate, subject to $f'''(b) - f'''(a) = 0$,

$$\epsilon \approx \frac{(b - a)^6}{1080n^6}[f^{\text{iv}}(b) - f^{\text{iv}}(a)] \ .$$

Applied to Part (a), this yields $n > 5.2$. For $n = 6$ Simpson's Rule does yield accurately to five places $\pi \approx 3.14159$.) For the integral of Part (a) the error is positive over part of the integral and negative over the rest; we have found only that the errors cancel and the result is much better than (14b) indicates. In Number 2 below, the estimate (ii) is directly useful.

2. Obtain $\log 3$ to four decimal place accuracy by numerical integration of $\int_1^3 \frac{1}{x}\,dx$.

Use $y''' = D^3(\frac{1}{x}) = -\frac{6}{x^4}$ in (ii) of the solution to Number (1c) to obtain the error estimate for Simpson's Rule,

$$\epsilon \approx \frac{6 \times 16}{180n^4}[1 - \frac{1}{3^4}] \approx \frac{8}{15n^4} ,$$

where we ignore the tern $\frac{1}{3^4}$. Impose the condition $\frac{8}{15n^4} < \frac{1}{2 \times 10^4}$, or

$$n > 10 \sqrt[4]{\frac{16}{15}} \approx 10.2 .$$

Since n must be even, take $n = 12$. (Note that (14b) would lead to $n = 18$.) For $x_k = 1 + \frac{k}{6}$, $(k = 0, 1, 2, \ldots, 12)$ obtain

$$\begin{array}{rl}
y_0 & = 1.000\,00 \\
y_{12} & \approx .333\,33 \\
\hline
a & = 1.333\,33 \\
y_1 & \approx .857\,14 \\
y_3 & \approx .666\,67 \\
y_5 & \approx .545\,45 \\
y_7 & \approx .400\,00 \\
y_{11} & \approx .352\,94 \\
\hline
b & = 3.283\,74 \\
4b & \approx 13.134\,96
\end{array}
\qquad\qquad
\begin{array}{rl}
y_2 & = .750\,00 \\
y_4 & = .600\,00 \\
y_6 & = .500\,00 \\
y_8 & = .428\,57 \\
y_{10} & = .375\,00 \\
\hline
c & = 2.653\,57 \\
2c & = 5.307\,14
\end{array}$$

$$\log 3 \approx \frac{1}{18}(a + 4b + 2c) = \frac{19.77543}{18}$$

$$\approx 1.09864 .$$

In fact, to five-place accuracy,

$$\log 3 = 1.09861 .$$

3. Estimate the integral

$$\int_0^{\pi/2} \frac{d\psi}{\sqrt{\cos\psi}}$$

of Section 13-1, Equation (1). (Hint: Compare Exercises 13-1, Number 2. Use the substitutions $\sin\psi = u^2$, $\cos\psi = 1 - \frac{v^2}{2}$ to obtain regular algebraic integrals.

Simpson's Rule cannot be applied to the integral as it stands because the integrand is not defined at $\psi = \frac{\pi}{2}$. From the observation in Exercises 3-1, Number 2 we have

$$I = \int_0^{\pi/2} \frac{d\psi}{\sqrt{\cos\psi}} = \int_0^{\pi/3} \frac{d\psi}{\sqrt{\cos\psi}} + \int_0^{\pi/6} \frac{d\psi}{\sqrt{\sin\psi}} .$$

Use the substitution $\sin\psi = u^2$ to obtain

(i) $$\int_0^{\pi/6} \frac{d\psi}{\sqrt{\sin\psi}} = 2\sqrt{2} \int_0^1 \frac{du}{\sqrt{4 - u^4}} ,$$

and apply $\cos\psi = 1 - \frac{v^2}{2}$ to obtain

(ii) $$\int_0^{\pi/3} \frac{d\psi}{\sqrt{\cos\psi}} = 2\sqrt{2} \int_0^1 \frac{dv}{\sqrt{(2 - v^2)(4 - v^2)}} .$$

We use Simpson's Rule with $n = 2$ for both integrals. For $y = \frac{1}{\sqrt{4 - u^4}}$ we have

$$y_0 = .500 , \ y_1 \approx .516 , \ y_2 \approx .577 ;$$

whence, $\int_0^1 \frac{du}{\sqrt{4 - u^4}} \approx .524$. For $z = \frac{1}{\sqrt{(2 - v^2)(4 - v^2)}}$, we have

$$z_0 \approx .356 , \ z_1 \approx .390 , \ z_2 \approx .577 ;$$

whence,

$$\int_0^1 \frac{dv}{\sqrt{(2 - v^2)(4 - v^2)}} \approx .416 .$$

Insert these results in (i) and (ii) and add to obtain

$$I \approx 2\sqrt{2} \times .940 \approx 2.66$$

which falls in the range indicated in the solution of Exercises 13-1, Number 2.

4. Verify that

$$\int_0^4 (x - 4)(x - 2)x \, dx = 0 .$$

Simpson's Rule is exact for cubic polynomials. Apply Simpson's Rule with $n = 1$.

5. Use the integral form of the Taylor remainder to obtain the error bounds for

(a) the Trapezoid Rule given in Formula (6),

With the remainders in the form given by Exercises 13-3, Number 9(a), we have instead of (3)

$$\int_{x_{k-1}}^{x_k} f(x)dx = y_{k-1}h + \frac{y'_{k-1}}{2} h^2 + \frac{1}{2} \int_{x_{k-1}}^{x_k} (x_k - x)^2 f''(x)dx$$

and, instead of (4),

$$\frac{h}{2}(y_{k-1} + y_k) = y_{k-1}h + \frac{y'_{k-1}}{2} h^2 + \frac{h}{2} \int_{x_{k-1}}^{x_k} (x_k - x)f''(x)dx .$$

These results yield (with the observation that $h = x_k - x_{k-1} \geq x_k - x$)

$$\text{(i)} \qquad |\epsilon_k| = \left| \int_{x_{k-1}}^{x_k} f''(x) \{\frac{h}{2}(x_k - x) - \frac{(x_k - x)^2}{2}\} dx \right|$$

$$\leq \frac{1}{2} \int_{x_{k-1}}^{x_k} |f''(x)| \{h(x_k - x) - (x_k - x)^2\} dx$$

$$\leq \frac{1}{2} \int_{x_{k-1}}^{x_k} M_2 \{h(x_k - x) - (x_k - x)^2\} dx$$

(formula continued)

$$\leq \frac{M_2}{2}\left\{-\frac{h(x_k - x)^2}{2} + \frac{(x_k - x)^3}{3}\right\}\Bigg|_{x_{k-1}}^{x_k}$$

$$\leq \frac{M_2}{2}\left\{\frac{h^3}{2} - \frac{h^3}{3}\right\}$$

$$\leq \frac{M_2}{12} h^3 .$$

(b) Simpson's Rule given by Formula (13).

Use the integral remainders in the derivation of (10) to obtain

$$\int_{x_{k-1}}^{x_{k+1}} f(x)dx = 2y_k h + \frac{y''_k}{3} h^3$$

$$+ \frac{1}{24}\int_{x_k}^{x_{k+1}} (x_{k+1} - x)^4 f^{iv}(x)dx$$

$$+ \frac{1}{24}\int_{x_k}^{x_{k-1}} (x_{k-1} - x)^4 f^{iv}(x)dx .$$

Similarly, with integral remainders (11) becomes

$$\frac{h}{3}[y_{k-1} + 4y_k + y_{k+1}] = 2y_k h + y''_k \frac{h^3}{3}$$

$$+ \frac{h}{18}\int_{x_k}^{x_{k+1}} (x_{k+1} - h)^3 f^{iv}(x)dx$$

$$+ \frac{h}{18}\int_{x_k}^{x_{k-1}} (x_{k-1} - x)^3 f^{iv}(x)dx .$$

For the error bound we then have

$$|\epsilon_k| = \left|\frac{1}{6}\int_{x_k}^{x_{k+1}} f^{iv}(x)\left\{\frac{h(x_k - x)^3}{3} - \frac{(x_k - x)^4}{4}\right\}dx\right.$$

$$\left. + \frac{1}{6}\int_{x_{k-1}}^{x_k} f^{iv}(x)\left\{\frac{h(x - x_{k-1})^4}{3} - \frac{(x_{k-1} - x)^5}{4}\right\}dx\right|$$

$$\leq \frac{M_4}{6} h^5[(\frac{1}{12} - \frac{1}{20}) + (\frac{1}{12} - \frac{1}{20})]$$

$$\leq \frac{M_4}{90} h^5 .$$

6. Using approximation to the integral $\int_1^n \log x \, dx$ obtain an inequality of the form

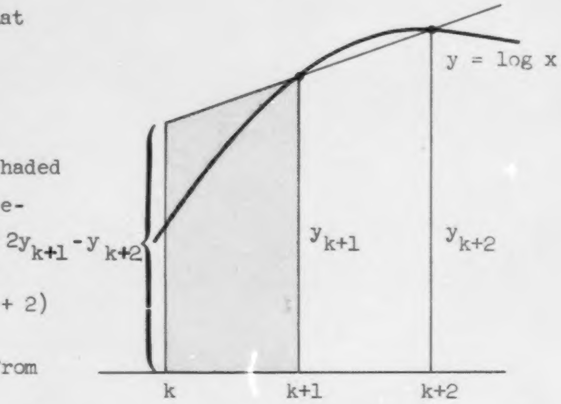
$$c\sqrt{n}\,(\tfrac{n}{e})^n \leq n!$$

(Hint: Note that the extension to the left of a chord to the graph $y = \log x$ lies above the curve.)

Following the hint, observe that

$$J_k = \int_k^{k+1} \log x \, dx$$

is less than the area of the shaded trapezoid in the figure. Consequently,



$$J_k \leq \tfrac{3}{2}\log(k+1) - \tfrac{1}{2}\log(k+2)$$

where $y_\nu = \log \nu$. Summing from 1 to $n - 1$ get

$$I_n = \sum_{k=1}^{n-1} J_k = \int_1^n \log x \, dx$$

$$\leq \tfrac{3}{2}\log 2 + \log 3 + \log 4 + \dots + \log n - \frac{\log(n+1)}{2}$$

$$\leq \log n! - \log\sqrt{\frac{n+1}{2}}\ .$$

Taking the value of I_n from (15), then obtain

$$n! \geq \frac{e}{\sqrt{2}}\sqrt{n+1}\,(\tfrac{n}{e})^n \geq \frac{e}{\sqrt{2}}\sqrt{n}\,(\tfrac{n}{e})^n\ .$$

7. Obtain asymptotic expressions for the following binomial coefficients.

(a) $\binom{2n}{n}$

$$\frac{2^{2n}}{\sqrt{\pi n}}$$

(b) $\binom{n}{k}$, k fixed.

$$\binom{n}{k} \sim \frac{\sqrt{2\pi n}\,(\frac{n}{e})^n}{k!\,\sqrt{2\pi(n-k)}\,(\frac{n-k}{e})^{n-k}}$$

$$\sim \frac{n^k}{k!e^k}\left(\frac{n}{n-k}\right)^{n-k}$$

$$\sim \frac{n^k}{k!e^k}\left(1+\frac{k}{n-k}\right)^{n-k} \sim \frac{n^k}{k!} .$$

(c) $\binom{pn}{n}$, for large p and n .

$$\binom{pn}{n} \sim \frac{\sqrt{2\pi pn}\,(\frac{pn}{e})^{pn}}{\sqrt{2\pi n}\,(\frac{n}{e})^n \cdot \sqrt{2\pi(p-1)n}\,[\frac{(p-1)n}{e}]^{(p-1)n}}$$

$$\sim \frac{p^n}{\sqrt{2\pi n}}\left(\frac{p}{p-1}\right)^{(p-1)n}$$

$$\sim \frac{p^n}{\sqrt{2\pi n}}\left[1+\frac{1}{p-1}\right]^{(p-1)n}$$

$$\sim \frac{(ep)^n}{\sqrt{2\pi n}}$$

8. Obtain an asymptotic expression for the coefficient of x^{2n+1} in the Taylor expansion of arc sin x (given in Example 13-3b).

Observe that the coefficient is given by

$$a_{2n+1} = \frac{(2n)!}{(2n+1)(n!)^2 2^{2n}} = \frac{1}{(2n+1)2^{2n}}\binom{2n}{n}$$

and use the solution of 7(a) to obtain

$$a_{2n+1} \sim \frac{1}{(2n+1)\sqrt{\pi n}} .$$

9. Obtain a sharper lower bound for $J_k - J_k^*$ than that of (19) and so improve the lower estimate for n! in (23). (Hint: Use the integral form for the Taylor remainder as in No. 5.)

From (i) in the solution of Number 5 with $f(x) = \log x$, we have

$$J_k - J_k^* = -\epsilon_k = \frac{1}{2}\int_k^{k+1} \frac{(k+1)(k+1-x) - (k+1-x)^2}{x^2}\,dx$$

$$> \frac{1}{2}\int_k^{k+1} \frac{(k+1)(k+1-x) - (k+1-x)^2}{(k+1)^2}\,dx$$

$$\geq \frac{1}{12(k+1)^2} .$$

From (20) it follows that

$$\log \frac{\lambda_\nu}{\lambda_{\nu+1}} > \frac{1}{12(\nu+1)^2} .$$

Now sum from n to $n + k - 1$ to obtain

$$\frac{\lambda_n}{\lambda_{n+k}} > \exp\{\frac{1}{12}\left[\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \dots + \frac{1}{(n+k)^2}\right]\} .$$

The sum in brackets is a Riemann upper sum for

$$\int_{n+1}^{n+k+1} \frac{1}{x^2}\,dx = \frac{1}{n+1} - \frac{1}{n+k+1} .$$

Thus $\lambda_n > \lambda_{n+k} \exp\{\frac{1}{12}[\frac{1}{n+1} - \frac{1}{n+k+1}]\}$. Now this result holds for all natural numbers k so that

$$\lambda_n \geq \lim_{k\to\infty} \lambda_{n+k} \exp\{\frac{1}{12}[\frac{1}{n+1} - \frac{1}{n+k+1}]\} ,$$

$$\geq \lambda \exp\{\frac{1}{12(n+1)}\}$$

hence

$$n! \geq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{1/12(n+1)} .$$

It may be of interest to compare this lower estimate and the upper estimate of (23) with the true value for some n . We have

$$17! = 355 , 687 , 428 , 096 , 000 .$$

Using a table of logarithms we obtain to five significant figures for $n = 17$, with $\alpha = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$,

$$\alpha = 3.5392 \times 10^{14}$$

$$\alpha e^{1/12(n+1)} = 3.5557 \times 10^{14}$$

$$\alpha e^{1/12(n-1)} = 3.5577 \times 10^{14} .$$

TC.13-5. Numerical Solution of First Order Differential Equations.

To obtain a useful degree of accuracy with Euler's method the number of steps is likely to be too high for convenient hand computation. For that reason, no computational exercises are provided. If an electronic computer is available to the class such exercises may easily be composed and assigned.

Solutions Exercises 13-5

1. Consider the solution by Euler's method of the initial value problem (2a,b) in a region where $f(x)$ and $g(y)$ have bounded derivatives. Obtain error estimates in the form of Equation (8).

We have, in the exact form of (3),

$$y_k = y_{k-1} + y'_{k-1}h + R_{2,k}$$

where $|R_{2,k}| \leq M_2 \frac{h^2}{2}$ and M_2 is an upper bound for y'' in the region of the hypothesis. Let A_0 and A_1 be upper bounds within the region for $f(x)$ and $f'(x)$, respectively, and B_0 and B_1 the bounds for $g(y)$ and $g'(y)$. Differentiate in (2a) to obtain

$$\begin{aligned} y'' &= f'(x)g(y) + f(x)g'(y)y' \\ &= f'(x)g(y) + f(x)^2 g'(y)g(y) ; \end{aligned}$$

whence, within the given region,

$$|y''| \leq A_1B_0 + A_0^2 B_1B_0 = K .$$

Thus obtain the exact form of (5),

$$\hat{y}_k - y_k = (\hat{y}_{k-1} - y_{k-1}) + hf(x_{k-1})[g(\hat{y}_{k-1}) - g(y_{k-1})] - R_{2,k} ,$$

where

$$|R_{2,k}| \leq K \frac{h^2}{2}$$

and, with $L = A_0B_1$,

$$|f(x_{k-1})[g(\hat{y}_{k-1}) - g(y_{k-1})]| \leq L(\hat{y}_{k-1} - y_{k-1}) .$$

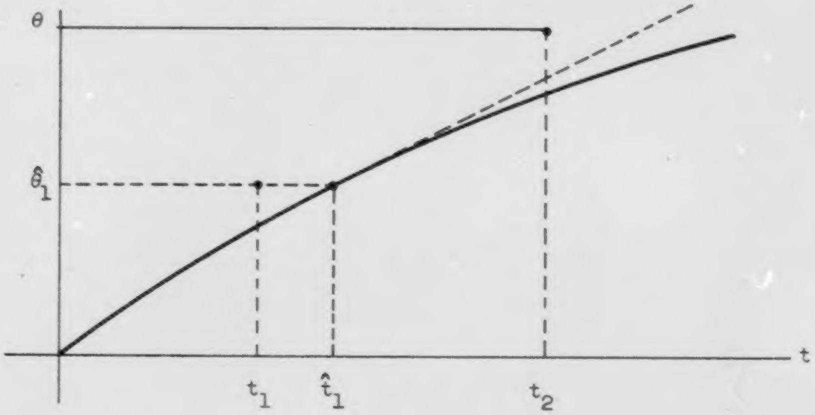
For $A = \max\{K,L\}$, Equations (7) and (8) follow.

2. Show for the initial value problem (9a,b) that the approximate solution is greater than the exact solution, namely that $\hat{\theta}_k > \theta_k$, $(k > 0)$.

Let the exact solution be $\theta = F(t)$. Since $\theta'' = -\frac{1}{2}\sin\theta$ the graph of the exact solution is flexed downward when $0 \leq \theta \leq \frac{\pi}{2}$. It follows for $0 < x \leq x_1$ that the graph lies below its tangent line at $x = 0$, namely, that

$$\hat{\theta} = \hat{\theta}_0 + t\sqrt{\cos\hat{\theta}_0} > \theta .$$

It follows for $t = h$, that $\hat{\theta}_1 > \theta_1$. Now, if $\hat{\theta}_1 \leq \frac{\pi}{2}$ let t_1 be the time when $F(\hat{t}_1) = \hat{\theta}_1$ (see figure).



Since $F(t)$ is increasing and $\hat{\theta}_1 > \theta_1$ it follows that $\hat{t}_1 > t_1 = h$. The graph $\theta = F(t)$ never lies above its tangent line at $\hat{t}_1$, hence

$$F(t) \leq \hat{\theta}_1 + (t - \hat{t}_1)\sqrt{\cos\hat{\theta}_1}$$

$$\leq \hat{\theta}_1 + (t - t_1)\sqrt{\cos\hat{\theta}_1} .$$

Take $t = t_2$ to obtain

$$\theta_2 = F(t_2) \leq \hat{\theta}_1 + h\sqrt{\cos\hat{\theta}_1} = \hat{\theta}_2 .$$

In exactly the same way it can be shown if $\theta_k < \hat{\theta}_k < \frac{\pi}{2}$ that $\theta_{k+1} < \hat{\theta}_{k+1}$.

3. **Show if Φ (x,y) is a function of x alone, that Euler's method for Equation (1) approximates the solution by successive Riemann sums.**

Let the differential equation and initial condition be

$$\frac{dy}{dx} = f(x)$$

and

$$y = y_0 \quad \text{at} \quad x = x_0 .$$

Then, from $\hat{y}_k = \hat{y}_{k-1} + h\, f(x_{k-1})$, obtain successively

$$\begin{aligned}\hat{y}_k &= \hat{y}_{k-2} + h[f(x_{k-2}) + f(x_{k-1})] \\ &= \hat{y}_{k-3} + h[f(x_{k-3}) + f(x_{k-2}) + f(x_{k-1})] \\ &= \dots \\ &= y_0 + \sum_{\nu=0}^{k-1} f(x_\nu)h \\ &= y_0 + \sum_{\nu=0}^{k-1} f(x_\nu)(x_{\nu+1} - x_\nu) .\end{aligned}$$

But $\sum_{\nu=0}^{k-1} f(x_\nu)(x_{\nu+1} - x_\nu)$ is a Riemann sum for $\int_{x_0}^{x_{k-1}} f(x)dx$.

Solutions Miscellaneous Exercises

1. (a) Obtain an iteration scheme for the zero of $f : x \to a - \frac{1}{x}$ and thus show how to calculate the reciprocal of a without divisions.

Apply Newton's method to obtain the scheme

$$a_{k+1} = a_k - \frac{f(a_k)}{f'(a_k)} = a_k(2 - aa_k) .$$

This scheme is especially practical for use on a digital computer since the computer routine for multiplication is much faster than that for division.

(b) Use the method obtained in (a) to calculate $\frac{1}{\pi}$ accurately to the extent indicated by the approximation $\pi \approx 3.141593$.

Take $a_0 = \frac{1}{3}$. Then

$$a_0 = \frac{1}{3}$$

$$a_1 = .3176 \qquad (2 - aa_0) \approx .9528$$

$$a_2 = .318308 \qquad (2 - aa_1) \approx 1.00223$$

$$a_3 = .318309865 \qquad (2 - aa_2) \approx 1.0000059$$

$$(2 - aa_3) \approx 1.0000000 .$$

Thus to seven-place accuracy

$$\frac{1}{\pi} = .3183099 .$$

2. In Section 13-2 we observed if $x_0 > 0$ is an approximation to $\sqrt{A}$ from one side, then $\frac{A}{x_0}$ is an estimate from the other side. We showed (for $A = 7$, but the proof is valid in general) that the arithmetic mean $x_1 = \frac{1}{2}(x_0 + \frac{A}{x_0})$ is an approximation from above. Show that the harmonic mean approximates $\sqrt{A}$ from below and estimate the error.

The harmonic mean is

$$y_1 = \frac{1}{\frac{1}{2}(\frac{1}{x_0} + \frac{x_0}{A})} = \frac{2Ax_0}{x_0^2 + A} .$$

Set $\epsilon_0 = x_0 - \sqrt{A}$. Then

$$\epsilon_1 = y_1 - \sqrt{A} = \frac{2A(\sqrt{A} + \epsilon_0)}{(\sqrt{A} + \epsilon_0)^2 + A} - \sqrt{A}$$

$$= - \frac{\epsilon_0^{\,2}\sqrt{A}}{(\sqrt{A} + \epsilon_0)^2 + A} \approx - \frac{\epsilon_0^{\,2}}{2\sqrt{A}} .$$

Thus $\epsilon_1 < 0$. The result can also be obtained from the observation that $y_1 = \frac{A}{x_1}$ where x_1 is the given arithmetic mean. (The natural attempt to improve the approximation by taking the arithmetic mean of the two therefore merely yields the second iterant, $\frac{1}{2}(x_1 + y_1) = \frac{1}{2}(x_1 + \frac{A}{x_1}) = x_2$.)

3. Compute $\int_0^{\pi} \frac{\sin x}{x}\, dx$ accurately to three decimal places.

For $f : x \to \frac{\sin x}{x}$ apply Simpson's Rule and use the error estimate (ii) from the solution of Exercises 13-4, Number 1(c), namely

$$\epsilon \approx \frac{\pi^4}{180n^4}[f'''(\pi) - f'''(0)] .$$

From

$$f'''(x) = \frac{1}{x^4}[-x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x]$$

we obtain $f'''(\pi) = \frac{\pi^3 - 6\pi}{\pi^4}$, and $f'''(0) = 0$. (It is understood here that $\frac{\sin x}{x}$ is extended continuously to $x = 0$.) Consequently,

$$\epsilon \approx \frac{\pi^3 - 6\pi}{180n^4} \approx \frac{.07}{n^4} .$$

We choose $n^4 > .07 \times 2 \times 10^3$ or $n > 3.5$. Thus we take $n = 4$ and obtain

$$\int_0^\pi \frac{\sin x}{x}\,dx \approx \frac{\pi}{12}\left[1 + \frac{4\sin\frac{\pi}{4}}{\frac{\pi}{4}} + \frac{2\sin\frac{\pi}{2}}{\frac{\pi}{2}} + \frac{4\sin\frac{3\pi}{4}}{\frac{3\pi}{4}}\right]$$

$$\approx \frac{1}{12}[\pi + 4 + \frac{32\sqrt{2}}{3}]$$

$$\approx 1.852 \ .$$

4. (a) Consider a right triangle with shorter side of length a, longer side b, and hypotenuse c. Let α be the angle opposite side a. Estimate the error in the approximation

$$\alpha \approx \frac{3a}{b + 2c} \ .$$

From $a = c \sin\alpha$, $b = c\cos\alpha$,

$$\hat{\alpha} = \frac{3a}{b + 2c} = \frac{3\sin\alpha}{2 + \cos\alpha} \ .$$

The obvious procedure is to expand in powers of α and use Taylor's Theorem. Instead we make use of estimates for the numerator and denominator (from Example 7-5b) to obtain

$$\frac{3\left(\alpha - \frac{\alpha^3}{6}\right)}{2 + \left(1 - \frac{\alpha^2}{2} + \frac{\alpha^4}{24}\right)} - \alpha \le \hat{\alpha} - \alpha \le \frac{3\left(\alpha - \frac{\alpha^3}{6} + \frac{\alpha^5}{120}\right)}{2 + \left(1 - \frac{\alpha^2}{2}\right)} - \alpha$$

or

$$-\frac{\alpha^5}{24\left(3 - \frac{\alpha^2}{2} + \frac{\alpha^4}{24}\right)} \le \hat{\alpha} - \alpha \le \frac{\alpha^5}{120\left(3 - \frac{\alpha^2}{2}\right)} \ .$$

Since α is the smallest angle in the triangle $0 < \alpha \le \frac{\pi}{4} < 1$. Consequently,

$$-\frac{\alpha^5}{24 \times 3} \le \hat{\alpha} - \alpha \le \frac{\alpha^5}{120(3 - \frac{1}{2})} \ .$$

whence,

$$|\hat{\alpha} - \alpha| \le \frac{\alpha^5}{72} \ .$$

For $\alpha = \frac{\pi}{4}$,

$$\frac{\alpha^5}{72} < \frac{0.3}{72} \le \frac{1}{240} < \tilde{.}0042 \ .$$

Thus the error never exceeds .004 radians or one quarter of a degree.

(b) Obtain an approximation for α in the form

$$\alpha \approx \frac{a(pb + qc)}{c^2} ,$$

where the constants p and q are so chosen that the error in the approximation is higher order than 3. Estimate the error.

Neglect terms of order higher than 3 to obtain

$$\begin{aligned} \hat{\alpha} &= \frac{a(pb + qc)}{c^2} = \sin \alpha \ (p \cos \alpha + q) \\ &\approx (\alpha - \frac{\alpha^3}{6})(p[1 - \frac{\alpha^2}{2}] + q) \\ &\approx (p + q)\alpha - (\frac{p}{2} + \frac{p + q}{6})\alpha^3 . \end{aligned}$$

Since this last expression is to be an exact expression for α, impose the conditions

$$p + q = 1 \ , \ \frac{p}{2} + \frac{p + q}{6} = 0$$

to get

$$p = -\frac{1}{3} \ , \ q = \frac{4}{3} .$$

Thus the desired formula is

$$\hat{\alpha} = \frac{a(4c - b)}{3c^2} .$$

To estimate the error observe that

$$\begin{aligned} \hat{\alpha} &= \frac{1}{3} \sin \alpha(4 - \cos \alpha) \\ &= \frac{4}{3} \sin \alpha - \frac{1}{6} \sin 2\alpha \end{aligned}$$

and use Taylor's Theorem to obtain

$$|\hat{\alpha} - \alpha| \leq \frac{M_5 \alpha^5}{120}$$

where M_5 is an upper bound for

$$\left|\frac{d^5\hat{\alpha}}{d\alpha^5}\right|$$

The fifth derivative is

$$\frac{d^5\alpha}{d\alpha^5} = \frac{4}{3}[\cos\alpha - 4\cos 2\alpha]$$

which is easily shown to reach its greatest absolute magnitude on $[0, \frac{\pi}{4}]$ at the endpoint $\alpha = \frac{\pi}{4}$. Consequently,

$$\left|\frac{d^5\hat{\alpha}}{d\alpha^5}\right| \le \left|\frac{4}{3}\left(\frac{1}{\sqrt{2}} - 4\right)\right| < 6 .$$

Take $M_5 = 6$, and find

$$|\hat{\alpha} - \alpha| < \frac{\alpha^5}{20} .$$

5. Consider the solid of revolution obtained by rotating the graph $y = f(x)$ of a nonnegative function $[a,b]$ about the x-axis. Let A_0, A_1, A_2 be the areas of the cross-sections of the solid perpendicular to the x-axis at $x = a$, $\frac{a+b}{2}$, b, respectively. Show that Simpson's Rule

$$V = \frac{b-a}{6}[A_0 + 4A_1 + A_2]$$

gives the exact volume for each of the following cases,

(a) frustrum of a right circular cone ($y = f(x)$ is a straight line),

(b) segment of a sphere ($y = f(x)$ is an arc of a circle with center on the x-axis),

(c) segment of paraboloid, ellipsoid or hyperboloid of revolution ($y = f(x)$ is an arc respectively of parabola, ellipse or hyperbola, with the x-axis as an axis of symmetry).

The volume is given by

$$V = \int_a^b \pi f(x)^2 dx = \int_a^b A(x)dx$$

where $A(x)$ is the cross-sectional area. In each case above $A(x)$ is a polynomial of degree three or less. Thus Simpson's Rule is

(a) $A(x) = \pi(\alpha x + \beta)^2$

(b) $A(x) = \pi[r^2 - (x - \alpha)^2]$

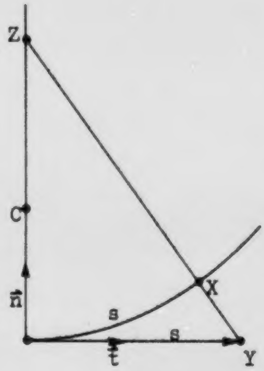
(c) $A(x) = \pi(x - \alpha)$, (paraboloid).

$A(x) = \pi b^2(1 - \frac{x^2}{a^2})$, (ellipsoid).

$A(x) = \pi b^2(\frac{x^2}{a^2} \pm 1)$, (hyperbola of one or two sheets).

The method also gives exact areas of the frustrum of any cone (In general a cone is the solid generated by the segments joining the points of a given region, the base, to a given point, the vertex. Thus tetrahedra and pyramids are cones.) where the cross-sections are taken along a line perpendicular to the base. However, we have given no general discussion of volume except for solids of revolution.

A6. Let $\vec{X} = \vec{r}(s)$ be the vectorial representation of a plane curve with arclength as parameter. Let $\vec{0} = \vec{r}(0)$. Set $\vec{Y} = s\vec{t}$ where $\vec{t}$ is the tangent at 0. Let Z be the point where the line XY meets the normal line through $\vec{0}$. Show if the curvature κ at 0 is not zero then to lowest order in s, $\vec{Z} = 3\vec{C}$ where C is the center of curvature.



Set $\kappa' = \left.\frac{d\kappa}{ds}\right|_{s=0}$. Then

$$\vec{X} = s\vec{t} + \frac{s^2}{2}\kappa\vec{n} + \frac{s^3}{6}(\kappa'\vec{n} - \kappa^2\vec{t}) + \dots$$

$$= \vec{Y} + \frac{s^2}{2}\kappa\vec{n} + \frac{s^3}{6}(\kappa'\vec{n} - \kappa^2\vec{t}) + \dots .$$

Thus,

$$\text{(i)} \qquad \vec{Z} = \kappa\vec{n} = s\vec{t} + \beta[\frac{s^2}{2}\kappa\vec{n} + \frac{s^3}{6}(\kappa'\vec{n} - \kappa^2 t)] + \dots ;$$

whence, for the coefficient of $\vec{t}$ in (i),

$$s - \frac{\beta\kappa^2 s^3}{6} = 0$$

and

$$\beta = \frac{6}{\kappa^2 s^2} .$$

Enter this value of β in (i) to obtain

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$$\vec{Z} = \frac{3}{\kappa}\,\vec{n} + s\,\frac{\kappa'}{\kappa^2}\,\vec{n} + \dots$$
$$\approx 3\vec{C}$$

to lowest order.

A7. Let a curve be given by $\vec{X} = \vec{r}(s)$ where s is arclength measured from $\vec{0} = \vec{r}(0)$. Consider any three distinct points $\vec{X}_1 = \vec{r}(s_1)$, $\vec{X}_2 = \vec{r}(s_2)$, $\vec{X}_3 = \vec{r}(s_3)$ where the s_i are confined to an δ-neighborhood of $s = 0$. Show that the circle through the three points approaches the osculating circle as δ approaches zero. (Assume the curvature at $s = 0$ is not zero.)

Let the circle through the three points be

$$(\vec{X} - \vec{A})^2 = r^2$$

where the center A and the radius r are to be determined. The center lies at the intersection of the perpendicular bisectors of the segments $\overline{X_1X_2}$ and $\overline{X_2X_3}$. Thus, for suitable values of the scalaro p and q,

$$\text{(i)} \qquad \vec{A} = \tfrac{1}{2}(\vec{X}_1 + \vec{X}_2) + p\vec{\nu} \times (\vec{X}_2 - \vec{X}_1)$$
$$= \tfrac{1}{2}(\vec{X}_2 + \vec{X}_3) + p\vec{\nu} \times (\vec{X}_3 - \vec{X}_2)$$

where $\vec{\nu}$ is the unit upward normal to the plane ($\vec{\nu} \times \vec{t} = \vec{n}$ and $\vec{\nu} \times \vec{n} = -\vec{t}$). Thus,

$$\vec{\nu} \times [p(\vec{X}_2 - \vec{X}_1) - q(\vec{X}_3 - \vec{X}_2)] = \tfrac{1}{2}(\vec{X}_3 - \vec{X}_1) .$$

To solve for p, take the dot product with $\vec{X}_3 - \vec{X}_2$ and obtain

$$\text{(ii)} \qquad p = \frac{(\vec{X}_3 - \vec{X}_2) \cdot (\vec{X}_3 - \vec{X}_1)}{2(\vec{X}_3 - \vec{X}_2) \cdot [\vec{\nu} \times (\vec{X}_2 - \vec{X}_1)]} .$$

Let $\vec{t}$, $\vec{n}$ and κ be respectively the tangent, normal, and curvature at $s = 0$. Use

$$\text{(iii)} \qquad \vec{X}_i = s_i\vec{t} + s_i^{\,2}\,\frac{\kappa}{2}\,\vec{n} + s_i^{\,3}[\alpha(s_i)\vec{t} + \beta(s_i)\vec{n}]$$

where $\alpha(s)$ and $\beta(s)$ are bounded. Observe from (iii) that

$$\text{(iv)} \quad \vec{X}_i - \vec{X}_j = (s_i - s_j)\vec{t} + (s_i^{\,2} - s_j^{\,2})\frac{\kappa}{2}\,\vec{n} + [s_i^{\,3}\alpha(s_i) - s_j^{\,3}\alpha(s_j)]\vec{t}$$
$$+ [s_i^{\,3}\beta(s_i) - s_j^{\,3}\beta(s_j)]\vec{n} .$$

Observe by the Law of the Mean that

$$\begin{aligned} s_i{}^3\alpha(s_i) - s_j{}^3\alpha(s_j) &= s_i{}^3[\alpha(s_j) + (s_j - s_i)\alpha'(u)] + s_j{}^3\alpha(s_j) \\ &= (s_i - s_j)[(s_i{}^2 + s_i s_j + s_j{}^2)\alpha(s_j) + s_i{}^3\alpha'(u)] \\ &= (s_i - s_j)a_{ij}\delta^2 \end{aligned}$$

where a_{ij} is bounded. Similarly,

$$s_i{}^3\beta(s_i) - s_j{}^3\beta(s_j) = (s_i - s_j)b_j\delta^2$$

where b_{ij} is bounded. Insert these results in (iv) to get

$$\vec{X}_i - \vec{X}_j = (s_i - s_j)\{(1 + a_{ij}\delta^2)\vec{t} + [(s_i + s_j)\frac{\kappa}{2} + b_{ij}\delta^2]\vec{n}\}$$

from which obtain the weaker estimate

$$\text{(v)} \qquad \vec{X}_i - \vec{X}_j = (s_i - s_j)\{(1 + a_{ij}\delta^2)\vec{t} + c_{ij}\delta\vec{n}\}$$

where c_{ij} is bounded. Use (v) in the numerator of (ii) and get

$$(\vec{X}_3 - \vec{X}_2)\cdot(\vec{X}_3 - \vec{X}_1) = (s_3 - s_2)(s_3 - s_1)[1 + \lambda\delta^2]$$

where λ is bounded. To estimate the denominator of (ii), Equation (v) will not be adequate. We must use (iii) and then extract the factor $(s_3 - s_2)(s_2 - s_1)(s_3 - s_1)$ from the denominator. For this purpose it is more convenient to take $\vec{X}_2$ as origin and rewrite (iii) in the form

$$\text{(vi)} \qquad \vec{X}_i = \vec{X}_2 + \sigma_i\vec{t}_2 + \sigma_i{}^2\frac{\kappa_2}{2}\vec{n}_2 + \sigma_1{}^3[\hat{\alpha}(\sigma_i)\vec{t}_2 + \hat{\beta}(\sigma_i)\vec{n}_2]$$

where $\sigma_i = s_i - s_2$. Also make use of

$$\text{(vii)} \qquad \left\{ \begin{aligned} \hat{\alpha}(\sigma_3) &= \hat{\alpha}(\sigma_1) + (\sigma_3 - \sigma_1)\hat{\alpha}'(u) \ , \\ \hat{\beta}(\sigma_3) &= \hat{\beta}(\sigma_1) + (\sigma_3 - \sigma_1)\hat{\beta}'(v) \ . \end{aligned} \right.$$

For brevity, write $\alpha_1 = \hat{\alpha}(\sigma_1)$, $\beta_1 = \beta(\sigma_1)$, $\alpha' = \hat{\alpha}'(u)$, and $\beta' = \hat{\beta}'(v)$. Then insert (vi) and (vii) in the denominator of (ii) and find, using $\sigma_i < 2\delta$

$$\begin{aligned} 2(\vec{X}_3 - \vec{X}_2)\cdot[\hat{v} \times (\vec{X}_2 - \vec{X}_1)] &= \sigma_1\sigma_3(\sigma_1 - \sigma_3)\{\kappa_2 - 2([\sigma_3 + \sigma_1]\beta_1 + \sigma_3{}^2\beta') \\ &\qquad + \kappa_2\sigma_1\sigma_3(\alpha_1 + \sigma_3\alpha') + 2\sigma_1{}^2\sigma_3{}^2(\alpha_1\beta' - \beta_1\alpha')\} \\ &= \sigma_1\sigma_3(\sigma_1 - \sigma_3)\{\kappa_2 + c\delta\} \end{aligned}$$

where c is bounded. Note that

$$\kappa_2 = \kappa + s_2 \frac{d\kappa}{ds}\bigg|_{s=\tau} = \kappa + d\delta$$

where d is bounded and τ lies between s_2 and 0. Consequently, the denominator has the form

$$(s_1 - s_2)(s_3 - s_2)(s_1 - s_3)[\kappa + \mu\delta]$$

where μ is bounded. Enter these results in (ii):

(viii) $$p = \frac{1 + \lambda\delta^2}{(s_2 - s_1)[\kappa + \mu\delta]} .$$

Now enter (viii) and (v) in the first expression for $\vec{A}$ in (i):

$$\vec{A} = \frac{1}{2}(\vec{X}_1 + \vec{X}_2) + \frac{1 + \lambda\delta^2}{\kappa + \mu\delta}[\vec{n} - c_{21}\delta t + a_{21}\delta^2\vec{n}] .$$

Note also that $\frac{1}{2}(\vec{X}_1 + \vec{X}_2) = \vec{0} + \delta\vec{U}$, where the vector $\vec{U}$ is bounded. Conclude that the center of the circle through the three points satisfies

$$\vec{A} = \frac{\vec{n}}{\kappa} + \delta\vec{V}$$

where V is bounded. Thus the limit of $\vec{A}$ as δ approaches zero is the center of the osculating circle. Furthermore, with $\vec{X}_1 = \delta\vec{W}$ where $\vec{W}$ is bounded, obtain for the radius

$$r^2 = |\vec{X}_1 - \vec{A}|^2 = |\delta(\vec{W} - \vec{V}) - \frac{\vec{n}}{\kappa}|^2$$
$$= \frac{1}{\kappa^2} - \delta[\frac{\vec{n} \cdot (\vec{W} - \vec{V})}{\kappa} - \delta(\vec{W} - \vec{V})^2]$$

from which conclude that the radius r has $\frac{1}{|\kappa|}$, the radius of the osculating circle, as a limit.

Teacher's Commentary

Chapter 14

SEQUENCES AND SERIES

TC14-1. Introduction.

The ideas of sequence and series used casually in the rest of the text are now the principal object of attention. The ground has been laid so that little time need be spent on the introductory concepts: the definition of limit (p.39) was framed to include disconnected domains: limits as "x approaches infinity" were introduced early (p. 231): the idea of limit for a sequence or series, without name, has been used frequently in the text (e.g., Sections 6-2, 7-5, 8-6, Example 10-6e, and throughout Chapter 13). For this reason it is possible to proceed directly to the basic existence theorems. For the existence proofs we need a form of the completeness axiom for the real numbers, and we use the Least Upper Bound Principle (Section A1-5) for this purpose.

The Cauchy inequality, Exercises A1-2, Number 16, (p. 253) is used in the exercises.

Mathematical induction (Section A3-1) is used in both text and exercises.

No attempt has been made to provide the usual routine practice exercises on the calculation of limits; it is presumed that the text has already given enough practice of this type.

TC14-2. Convergence of Sequences.

There are some conventions concerning sequences and series which we have made use of without explicit mention in the text. For example, the definition of sequence in Section 14-1 requires that the index set be the entire set of natural numbers. For many purposes it is convenient to extend the definition of sequence to functions $a : k \to a_k$ where the domain is of all integers from a certain integer ν onward, $\{k : k \geq \nu\}$.

In particular, it is often convenient to take $\nu = 0$. Of course, it is possible to relate a directly to a sequence b , defined strictly, namely $b : k \to a_{\nu+k-1}$.

In a similar vein, we note that the function $k \to \frac{1}{a_k}$ which appears in the statement of Theorem 14-2e is not a sequence if any of the terms a_k happens to be zero. However, since there is some integer ν for which $k \geq \nu$ implies $a_k \neq 0$, this fact is irrelevant for the conclusion of the theorem. Similarly, in the Corollary to Theorem 14-2e, the function $k \to \frac{a_k}{b_k}$ will not be a sequence if $b_k = 0$ for any k .

Finally, we adopt a convention used tacitly on a few occasions (e.g., the proof of Theorem 14-5b), for any integer j not in the domain of $a : k \to a_k$, that $a_j = 0$. Thus, for a sequence a , strictly defined, in the proof of Theorem 14-5b, for a given positive r when $a_1 \geq r$ it will not be possible to find a natural number index i_1 such that

$$\sum_{k=1}^{i_1} a_k^{+} < r \leq \sum_{k=1}^{i_1+1} a_k^{+} .$$

We can handle this situation by extending the summation to include $k = 0$ and using the index. In addition it is convenient to adopt the convention that an empty sum is zero; this, too, is a way of handling the difficulty in the preceding example.

Solutions Exercises 14-2

1. Prove

(a)	Theorem 14-2a	(h)	Lemma 14-2
(b)	Theorem 14-2b	(i)	Corollary 1 to Lemma 14-2
(c)	Theorem 14-2c	(j)	Corollary 2 to Lemma 14-2
(d)	Theorem 14-2d	(k)	Corollary to Theorem 14-2e
(e)	Theorem 14-2e	(ℓ)	Corollary 1 to Theorem 14-2f
(f)	Theorem 14-2f	(m)	Corollary 2 to Theorem 14-2f
(g)	Corollary to Theorem 14-2c		

Parallel the proofs of Chapter 3.

2. Show that if $A_1 < A < A_2$, where $A = \lim_{k\sim\infty} a_k$, then there is a number ω such that $k > \omega$ implies $A_1 < a_k < A_2$.

Take $\epsilon = \min\{A - A_1 , A_2 - A\}$ and $\omega = \Omega(\epsilon)$. Then for $k > \omega$, $|A - a_k| < \epsilon$ or $A - \epsilon < a_k < A + \epsilon$. But $A_1 < A - \epsilon < a_k < A + \epsilon < A_2$.

3. Prove that $\lim_{k\sim\infty} |a_k| = 0$ if, and only if $\lim_{k\sim\infty} a_k = 0$.

Use $||a_k| - 0| = |a_k - 0|$.

4. Let f be a function whose domain contains the point a and points of every deleted neighborhood of a . Prove the converse of Theorem 14-2g. Namely, if $\lim_{n\sim\infty} f(x_n) = f(a)$ for every sequence $n \to x_n$ whose terms lie in the domain of f , and which has the limit a , then f is continuous at a .

There exists a sequence x , with values in the domain of f and $\lim_{n\sim\infty} x_n = a$, since every neighborhood of a contains points in the domain of f . Suppose f is not continuous at a . On the basis of this supposition we shall construct a sequence $x : n \to x_n$ converging to a , but the sequence $n \to f(x_n)$ will not converge to $f(a)$. Since f is not continuous at a there exists an $\epsilon > 0$ such that for each $\delta > 0$ there exists a point $\xi(\delta)$ satisfying

$$|\xi(\delta) - a| < \delta \text{ and } |f(\xi(\delta)) - f(a)| \geq \epsilon .$$

Thus the sequence $n \to x_n = \xi(\frac{1}{n})$ satisfies

$$|x_n - a| < \frac{1}{n} \quad \text{and} \quad |f(x_n) - f(a)| \geq \epsilon .$$

Hence $\lim_{n \to \infty} x_n = a$ while $n \to f(x_n)$ does not converge to $f(a)$.

5. Find $\lim_{n \to \infty} (\sqrt{n^2 + n} - n)$.

$$\sqrt{n^2 + n} - n = \frac{n^2 + n - n^2}{\sqrt{n^2 + n} + n} = \frac{1}{\sqrt{1 + \frac{1}{n}} + 1}$$ which converges to $\frac{1}{2}$.

6. Find the limits of the following sequences

(a) $n \to (1 + \frac{1}{n^2})^n$;

Since $(1 + \frac{1}{n})^n$ converges to $e < 3$, (Example 14-2e) for n sufficiently large $(1 + \frac{1}{n^2})^{n^2} < 3$. But,

$1 < (1 + \frac{1}{n^2})^n = [(1 + \frac{1}{n^2})^{n^2}]^{1/n} < 3^{1/n}$ which converges to 1 (Example 14-2c). Thus, by the Squeeze Theorem, $\lim_{n \to \infty} (1 + \frac{1}{n^2})^n = 1$.

(b) $n \to \frac{r^n}{n!}$;

Take $k > 2r$, then for $n > k$,

$$\frac{r^n}{n!} = \frac{r^k}{k!} \cdot \frac{r}{k+1} \cdot \frac{r}{k+2} \cdot \ldots \cdot \frac{r}{n} < \frac{r^k}{k!} \cdot \frac{1}{2^{n-k}} .$$

Thus $\lim_{n \to \infty} \frac{r^n}{n!} = 0$ by the Squeeze Theorem and $\lim_{k \to \infty} \frac{1}{2^k} = 0$ (Example 14-2b).

(c) $\dfrac{1}{n^{\alpha}}$, $\alpha > 0$;

Given $\epsilon > 0$, let $\omega = (\frac{1}{\epsilon})^{1/\alpha}$. Then if $n > \omega$, we have $\dfrac{1}{n^{\alpha}} < \epsilon$.

(d) $n \to \dfrac{\log n}{n^{\alpha}}$, $\alpha > 0$.

For $\beta > 0$,

$$\log n = \int_1^n \frac{dt}{t} < \int_1^n \frac{dt}{t^{1-\beta}} \leq \frac{t^{\beta}}{\beta}\Big|_1^n \leq \frac{n^{\beta}}{\beta} - \frac{1}{\beta} .$$

Thus, choosing $0 < \beta < \alpha$, we have

$$0 \leq \frac{\log n}{n^{\alpha}} < \frac{1}{\beta n^{\alpha-\beta}} - \frac{1}{\beta n^{\alpha}} ,$$

which converges to 0 by Part (c). Thus

$$\lim_{n \to \infty} \frac{\log n}{n^{\alpha}} = 0 .$$

7. Show that $\sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots}}} = 2$; that is, show that the sequence $a : k \to a_k$ defined by $a_1 = \sqrt{2}$, $a_{k+1} = \sqrt{2 + a_k}$ converges and the limit is 2 .

By induction show that $a_n \leq 2$ for all n : $a_1 = 2$ and $a_k \leq 2$ implies that

$$a_{k+1} = \sqrt{2 + a_k} \leq \sqrt{4} \leq 2 .$$

On the other hand since

(i) $$a_{k+1}^2 = 2 + a_k$$

and $a_k \leq 2$ for all k , we have

$$2a_{k+1} \geq a_{k+1}^2 \geq 2 + a_k \geq a_k + a_k \geq 2a_k ;$$

whence a is an increasing sequence. It follows that a must converge, say, to r .

Then, apply the elementary limit theorems to (i) and get $r^2 = 2 + r$, hence $r = 2$ or $r = -1$. Since r must be positive

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \dots}}} = 2 .$$

Alternative Solution: Apply the methods of Section 13-2 to the iteration scheme $a_{k+1} = \sqrt{2 + a_k}$ for approximation to 2.

8. Show that $\sqrt{2\sqrt{2\sqrt{2 \dots}}} = 2$, namely, that the sequence $a : k \to a_k$ defined by $a_1 = \sqrt{2}$, $a_{k+1} = \sqrt{2a_k}$ converges and the limit is 2.

$$\begin{aligned} a_k &= 2^{1/2} \cdot 2^{1/4} \dots 2^{1/2^k} = 2^{1/2 + 1/4 + \dots + 1/2^k} \\ &< 2^{1/2 + 1/4 + \dots + 1/2^k}\, 2^{1/2^{k+1}} , \end{aligned}$$

that is,

$$a_k < a_{k+1} < 2^{\sum_{k=1}^{\infty} (\frac{1}{2})^k} \leq 2^1 \leq 2 .$$

Hence a is a monotone sequence bounded by 2 and therefore convergent, say, to r. Since $r^2 = 2r$, $r = 2$.

Alternative Solution: Apply the methods of Section 13-2 to the iteration scheme

$$a_{k+1} = \sqrt{2a_k}$$

for approximation to 2.

9. The Fibonacci numbers are defined by $f_0 = f_1 = 1$ and $f_{n+2} = f_n + f_{n+1}$, $n = 0, 1, 2, \dots$. Find $\lim\limits_{n \to \infty} \frac{f_{n+1}}{f_n}$.

Suppose first that the limit exists and set $r = \lim\limits_{n \to \infty} \frac{f_{n+1}}{f_n}$. Since

$$\text{(i)} \qquad \frac{f_{n+2}}{f_{n+1}} = \frac{1}{\dfrac{f_{n+1}}{f_n}} + 1 ,$$

r is a solution of $r^2 = 1 + r$. Thus $r = \frac{1 \pm \sqrt{5}}{2}$. Since r must be positive $r = \frac{1 + \sqrt{5}}{2}$. It remains to be shown that the sequence $n \to r_n = \frac{f_{n+1}}{f_n}$ converges. Now, write (i) in the form

$$\text{(ii)} \qquad r_{n+1} = \frac{1}{r_n} + 1 .$$

Since f is increasing $r_n > 1$, hence, from (ii) with $r_{n+1} > 1$, $r_n < 2$. Furthermore, since $r_n < 2$, we have again from (ii) $r_{n+1} > \frac{3}{2}$. Thus, $\frac{3}{2} < r_n < 2$ for $n \geq 1$. Hence

$$|r_{n+1} - r_n| = |(1 + \frac{1}{r_n}) - (1 + \frac{1}{r_{n-1}})| = |\frac{1}{u^2}||r_n - r_{n-1}|$$

where u is between $\frac{3}{2}$ and 2. Thus $|r_{n+1} - r_n| < \frac{4}{9}|r_n - r_{n-1}|$, and by induction $|r_{n+1} - r_n| < (\frac{4}{9})^{n-1}|r_2 - r_1| < (\frac{4}{9})^{n-1} < (\frac{1}{2})^{n-1}$;

$$\begin{aligned} |r_{n+k} - r_n| &= |r_{n+k} - r_{n+k-1} + r_{n+k-1} - r_{n+k-2} + \dots + r_{n+1} - r_n| \\ &\leq |r_{n+k} - r_{n+k-1}| + \dots + |r_{n+1} - r_n| \\ &< \sum_{j=n}^{k-1} (\tfrac{1}{2})^{j-1} < \frac{1}{2^{n-1}} . \end{aligned}$$

Thus $n \to r_n$ is a Cauchy sequence and therefore convergent.

__Alternative Solution__: Apply the methods of Section 13-2 to the iteration scheme (ii) for $r = \frac{1 + \sqrt{5}}{2}$.

10. Show that the sequence

$$a : n \to a_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log n$$

converges. (The limit of this sequence is called Euler's constant, γ. It is not known whether or not γ is rational.)

$$\begin{aligned} a_n &= \sum_{k=1}^{n} \frac{1}{k} - \int_1^n \frac{dt}{t} = 1 + \sum_{k=2}^{n} \left[\frac{1}{k} - \int_{k-1}^{k} \frac{dt}{t}\right] \\ &= 1 + \sum_{k=2}^{n} \int_{k-1}^{k} (\frac{1}{k} - \frac{1}{t})dt > 1 + \sum_{k=2}^{n+1} \int_{k-1}^{k} (\frac{1}{k} - \frac{1}{t})dt , \end{aligned}$$

since $\int_{k-1}^{k} (\frac{1}{k} - \frac{1}{t})dt$ is negative. Thus, since $a_n > a_{n+1}$, a is a decreasing sequence. Furthermore, $1 + \frac{1}{2} + \ldots + \frac{1}{n-1}$ is an upper sum for $\int_1^n \frac{dt}{t}$ and is therefore greater than $\log n$; hence $a_n > 0$ for all n . Consequently, a converges by the Monotone Convergence Theorem (Theorem 14-2h).

11. Given a sequence $a : n \to a_n$, form the sequence

$$\sigma : n \to \sigma_n = \frac{1}{n} \sum_{k=1}^{n} a_k \, .$$

(a) Prove that if $\lim_{n\sim\infty} a_n = m$ then $\lim_{n\sim\infty} \sigma_n = m$.

Given $\epsilon > 0$, there exists an integer $\nu = N(\epsilon)$, such that if $n > \nu$ then $|a_n - m| < \epsilon$. Let $k = \nu + 1$ and set

$$\omega = \max\{2k \, , \frac{2}{\epsilon} \sum_{i=1}^{k} a_i + k|m|\} \, .$$

Then for $\ell > \omega$ we have

$$|\sigma_\ell - m| = \left| \frac{\sum_{i=1}^{k} a_i + \sum_{i=k+1}^{\ell} (a_i - m) + -km}{\ell} \right|$$

$$\leq \frac{\sum_{i=1}^{k} |a_i|}{\ell} + \frac{(\ell - k)\epsilon}{\ell} + \left|\frac{km}{\ell}\right|$$

$$\leq \frac{\sum_{i=1}^{k} |a_i|}{\ell} + \frac{\epsilon}{2} + \frac{k|m|}{\ell} < \epsilon \, .$$

(b) Show that σ may converge while a does not.

Let $a : n \to a_n = 1 + (-1)^n$. Then

$$\sigma_n = \frac{1}{n} \sum_{k=1}^{n} (1 + (-1)^k) = \begin{cases} \frac{1}{2} & \text{if } n \text{ is even} \\ \frac{1}{2} - \frac{1}{n} & \text{if } n \text{ is odd.} \end{cases}$$

Thus $\lim\limits_{n \sim \infty} \sigma_n = \frac{1}{2}$ while a does not converge.

12. Prove that if $c : k \to c_k$ is a subsequence of $b : k \to b_k$, then b is a subsequence of $a : k \to a_k$, then c is a subsequence of a .

There exist increasing sequences of positive integers

$$\mu : k \to \mu_k \quad \text{and} \quad \nu : k \to \nu_k$$

such that

$$b_k = a_{\mu_k} \quad \text{and} \quad c_k = b_{\nu_k} ,$$

hence $c_k = a_{(\mu\nu)_k}$. But the composition

$$\mu\nu : k \to (\mu\nu)_k$$

of increasing functions is increasing, hence c is a subsequence of a .

13. Find a sequence with no convergent subsequence.

$n \to n$ or any sequence that does not have a bounded subsequence.

14. Show that if $k \to a_{i_k}$ is a subsequence of $k \to a_k$ then $i_k \geq k$.

From the definition of subsequence, $i : k \to i_k$ is an increasing sequence of positive integers. Hence $i_1 \geq 1$. Proceed by induction: if $i_k \geq k$ then $i_{k+1} > i_k$ implies

$$i_{k+1} \geq i_k + 1 \geq k + 1 .$$

15. Show that if $k \to s_{i_k}$ and $k \to s_{j_k}$ are two subsequences of $n \to s_n$ satisfying $\lim_{k\to\infty} s_{i_k} = \lim_{k\to\infty} s_{j_k} = S$ and the sets of indices i_k and j_k together include all natural numbers, then $\lim_{k\to\infty} s_k = S$.

To any given $\epsilon > 0$ we can associate integers ν_1 and ν_2 , such that $|S - s_{i_k}| < \epsilon$ if $k > \nu_1$ and $|S - s_{j_k}| < \epsilon$ if $k > \nu_2$. Hence if $k > \max(\nu_1 , \nu_2)$ then $|S - s_k| < \epsilon$.

16. Let $a : n \to a_n$ be a bounded sequence. Let C be the set of limits of subsequences of a . (The elements of C are called cluster points of a .) The least upper bound of C , $\sup C$, is called the limit superior of a and is written $\overline{\lim}\, a_n$. Prove that $\overline{\lim}\, a_n \in C$.

To each n there is a subsequence $k \to a_{n_k}$ converging to a cluster point ℓ_n where $\overline{\lim}\, a_n - \frac{1}{n} \le \ell_n \le \overline{\lim}\, a_n$. The subsequence $k \to a_{k_k}$ converges to $\sup a$.

17. As is in Exercise 16 define the limit inferior of a as $\underline{\lim}\, a_n = \inf C$, where $\inf C$ is the greatest lower bound or infinum of C . Prove $\underline{\lim}\, a_n \in C$.

Set $b = -a$. Then $\underline{\lim}\, a_n = -\overline{\lim}\, b_n$.

18. For each of the following sequences a , find $\overline{\lim}\, a_n$ and $\underline{\lim}\, a_n$.

(a) $a : n \to (-1)^n$

$\overline{\lim}\, a_n = 1$, $\underline{\lim}\, a_n = -1$

(b) $a : n \to \cos \frac{2n\pi}{5}$

$\overline{\lim}\, a_n = 1$, $\underline{\lim}\, a_n = \cos \frac{4\pi}{5}$

(c) $a : n \to \frac{1}{n}$

$\overline{\lim}\, a_n = \underline{\lim}\, a_n = 0$

(d) $a : n \to \alpha + (1 + (-1)^n)(\frac{\beta - \alpha}{2})$

$$\overline{\lim}\, a_n = \max(\alpha,\beta) \ , \ \underline{\lim}\, a_n = \min(\alpha,\beta)$$

19. Let $a : n \to a_n$ be bounded, $|a_n| < M$. Suppose that

$$A_1 < \underline{\lim}\, a_n \leq \overline{\lim}\, a_n < A_2 \ .$$

Prove that there exists an ω such that for $k > \omega$,

$$A_1 < a_k < A_2 \ .$$

If no such ω exists with the stated property then there exists a subsequence $b : i \to b_i$ satisfying either

$$-M \leq b_i \leq A_1 \quad \text{or} \quad M \geq b_i \geq A_2$$

for all i . Hence, b has a convergent subsequence c (which is then also a convergent subsequence of a) whose limit C satisfies either

$$C \leq A_1 < \underline{\lim}\, a_n \quad \text{or} \quad C \geq A_2 > \overline{\lim}\, a_n$$

of which neither is possible.

20. Suppose that a number A is less than the limit superior of the bounded sequence $a : n \to a_n$, that is, $A < \overline{\lim}\, a_n$. Show that a has a subsequence $b : k \to b_k = a_{i_k}$ satisfying $b_k > A$ for all k .

Since $A < \overline{\lim}\, a_n$ there is a subsequence, $c : k \to c_k = a_{j_k}$, of a whose limit C is greater than A . Thus there exists an integer ν , such that for $k > \nu$

$$|c_k - C| < C - A \quad \text{or} \quad c_k > A \quad \text{for} \quad k > \nu \ .$$

Consequently the terms of the sequence defined by $b : k \to b_k = c_{i_{k+\nu}}$ are greater than A .

21. Let f be continuously differentiable and consider the sequences $n \to a_n$ and $n \to b_n$ which both converge to a , where $a_n \neq b_n$ for $n = 1 , 2 , \ldots$. Show that the sequence

$$n \to \frac{f(a_n) - f(b_n)}{a_n - b_n}$$

converges to $f'(a)$.

By the Mean Value Theorem there exists c_n between a_n and b_n such that

$$\frac{f(a_n) - f(b_n)}{a_n - b_n} = f'(c_n) .$$

Thus by the Corollary to the Squeeze Theorem $\lim_{n\sim\infty} c_n = a$, and by Theorem 14-2g for f' , $\lim_{n\sim\infty} f'(c_n) = f'(a)$.

∧22. Show by an example that the continuity of the derivative is essential in Number 21.

Let $f : x \to x^2 \sin \frac{1}{x}$ if $x \neq 0$ and $f(0) = 0$. Let $a_n = [(4n - 1)\frac{\pi}{2}]^{-1}$ and $b_n = [(4n + 1)\frac{\pi}{2}]^{-1}$. Then $f'(0) = 0$, $\lim_{n\sim\infty} a_n = \lim_{n\sim\infty} b_n = 0$, and $f(a_n) = a_n^2$ while $f(b_n) = -b_n^2$. Hence

$$\frac{f(a_n) - f(b_n)}{a_n - b_n} = \frac{a_n^2 + b_n^2}{a_n - b_n} = \frac{2}{\pi} \frac{|6n^2 + 1|}{|6n^2 - 1|}$$

which converges to $\frac{2}{\pi}$.

Solutions Exercises 14-3

1. Test the following series for convergence.

(a) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n}$

$\frac{\sqrt{n+1} - \sqrt{n}}{n} = \frac{1}{(\sqrt{n+1} + \sqrt{n})n} < \frac{1}{n^{3/2}}$. Thus $\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n}$ converges by the first comparison test and the p-test.

(b) $\sum_{n=3}^{\infty} \frac{1}{n(\log n)(\log \log n)^2}$

$$\int_3^\infty \frac{dt}{t(\log t)(\log\log t)^2} = \frac{-1}{\log\log t}\Bigg|_3^\infty = \frac{1}{\log\log 3}\ .$$

Thus $\sum\limits_{n=1}^{\infty} \frac{1}{n(\log n)(\log\log n)^2}$ converges by the Integral Test.

(c) $\sum\limits_{n=1}^{\infty} \frac{(n+1)2^n}{n3^n}$

Apply the Root Test: $\lim\limits_{n\sim\infty} \sqrt[n]{\frac{n+1}{n}(\frac{2}{3})^n} = \frac{2}{3} \lim\limits_{n\sim\infty} \sqrt[n]{\frac{n+1}{n}} = \frac{2}{3}$;

thus $\sum\limits_{n=1}^{\infty} \frac{(n+1)^n}{n3^n}$ converges.

(d) $\sum\limits_{n=1}^{\infty} \frac{n^3}{n!}$

Apply the Ratio Test: $\lim\limits_{n\sim\infty} \frac{(n+1)^3}{(n+1)!} \; \frac{n!}{n^3} = \lim\limits_{n\sim\infty} \frac{(n+1)^2}{n^3} = 0$;

thus $\sum\limits_{n=1}^{\infty} \frac{n^3}{n!}$ converges.

2. Does the series $\sum\limits_{n=1}^{\infty} (\frac{n}{n+1})^{n^2}$ converge?

Yes, since

$$\lim_{n\sim\infty} \sqrt[n]{(\frac{n}{n+1})^{n^2}} = \lim_{n\sim\infty} (\frac{n}{n+1})^n = \lim_{n\sim\infty} \frac{1}{(1+\frac{1}{n})^n} = \frac{1}{e} < 1\ .$$

3. Find a suitable $\omega = \Omega(\epsilon)$ for each of the following series.

(a) $\sum_{n=2}^{\infty} \frac{1}{n \log^2 n}$

$$\left| \sum_{n=2}^{\infty} \frac{1}{n \log^2 n} - \sum_{n=2}^{k} \frac{1}{n \log^2 n} \right| = \left| \sum_{n=k+1}^{\infty} \frac{1}{n \log^2 n} \right|$$

$$\leq \int_k^{\infty} \frac{dx}{x \log^2 x} \leq - \frac{1}{\log x} \Big|_k^{\infty} \leq \frac{1}{\log k} .$$

Hence

$$\left| \sum_{n=2}^{\infty} \frac{1}{n \log^2 n} - \sum_{n=2}^{k} \frac{1}{n \log^2 n} \right| < \epsilon$$

if $\frac{1}{\log k} < \epsilon$, that is, if $k > e^{1/e}$. Hence we may take $\Omega(\epsilon) = e^{1/e}$.

(b) $\sum_{n=0}^{\infty} \frac{n}{3^n}$

Since $\frac{n+1}{3^{n+1}} \cdot \frac{3^n}{n} = \frac{1}{3} \frac{n+1}{n} \leq \frac{2}{3}$ for all n, hence, as in the proof of the Ratio Test, $\frac{n}{3^n} \leq (\frac{2}{3})^n$ for all n. Therefore

$$\left| \sum_{n=0}^{\infty} \frac{n}{3^n} - \sum_{n=0}^{k} \frac{n}{3^n} \right| = \left| \sum_{n=k+1}^{\infty} \frac{n}{3^n} \right|$$

$$\leq \sum_{n=k+1}^{\infty} (\tfrac{2}{3})^n \leq (\tfrac{2}{3})^{k+1} \sum_{n=0}^{\infty} (\tfrac{2}{3})^n$$

$$\leq (\tfrac{2}{3})^{k+1} \frac{1}{1 - \frac{2}{3}} \leq 3(\tfrac{2}{3})^{k+1}$$

$$\leq 2(\tfrac{2}{3})^k$$

which is less than ϵ if $k > \frac{\log \frac{\epsilon}{2}}{\log \frac{2}{3}}$. Hence we may take

$$\Omega(\epsilon) = \frac{\log \frac{\epsilon}{2}}{\log \frac{2}{3}} .$$

(c) $\sum_{n=1}^{\infty} \frac{\sqrt{n-1}}{n^2}$

$\frac{\sqrt{n-1}}{n^2} < \frac{1}{n^{3/2}}$. Hence

$$\left|\sum_{n=1}^{\infty} \frac{\sqrt{n-1}}{n^2} - \sum_{n=1}^{k} \frac{\sqrt{n-1}}{n^2}\right| = \sum_{n=k+1}^{\infty} \frac{\sqrt{n-1}}{n^2}$$

$$< \sum_{n=k+1}^{\infty} \frac{1}{n^{3/2}} < \int_k^{\infty} \frac{dt}{t^{3/2}}$$

$$\leq \frac{-2}{\sqrt{t}}\Big|_k^{\infty} \leq \frac{2}{\sqrt{k}}$$

which is less than ϵ if $k > (\frac{2}{\epsilon})^2$. Hence we may take
$\Omega(\epsilon) = (\frac{2}{\epsilon})^2$.

4. Show that if $a_n \geq 0$, $n = 1, 2, \ldots$, and $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges.

Let $s_n = \sum_{k=1}^{n} a_k$ and $t_n = \sum_{k=1}^{n} \frac{\sqrt{a_k}}{k}$. Then by the Cauchy inequality (Exercises A1-2, No. 16)

$$t_n^{\,2} = \left(\sum_{k=1}^{n} \frac{\sqrt{a_k}}{k} \right)^2 \leq \sum_{k=1}^{n} a_k \sum_{k=1}^{n} \frac{1}{k^2} \leq \sum_{k=1}^{\infty} a_k \sum_{k=1}^{\infty} \frac{1}{k^2} .$$

Thus the sequence $n \rightarrow t_n$ is bounded and monotone increasing, therefore convergent.

5. Let $a : k \rightarrow a_k$ be a monotone decreasing sequence. Show that if a has a subsequence $k \rightarrow a_{i_k}$ for which $a_{i_k} > \frac{1}{i_k}$ then $\sum_{k=1}^{\infty} a_k$ diverges.

Given any n, choose $i_k > 2n$. Then $\sum_{j=n}^{i_k} a_j \geq \sum_{j=n}^{i_k} \frac{1}{i_k} \geq \sum_{j=n}^{2n} \frac{1}{2n} \geq \frac{1}{2}$.

Thus $\sum_{j=1}^{\infty} a_j$ diverges by the Cauchy criterion for series.

6. Show that if the series of positive terms $\sum_{i=1}^{\infty} a_i$ diverges then $\sum_{i=1}^{\infty} \frac{a_i}{s_i}$ diverges, where $s_i = \sum_{k=1}^{i} a_k$.

Since $s : n \rightarrow s_n$ is unbounded we can associate with each n an integer ν so that $2s_n < s_{n+\nu}$. Hence

$$\sum_{i=n+1}^{n+\nu} \frac{a_i}{s_i} > \frac{1}{s_{n+\nu}} \sum_{i=n+1}^{n+\nu} a_i \geq \frac{s_{n+\nu} - s_n}{s_n + \nu} \geq 1 - \frac{s_n}{s_{n+\nu}} > \frac{1}{2} .$$

Hence $\sum_{i=1}^{\infty} \frac{a_i}{s_i}$ diverges by the Cauchy criterion for series.

7. Show that if the series of positive terms $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} a_n^{\,2}$ converges.

By the n-th term test, for n sufficiently large $a_n < 1$. Thus

Thus $a_n^2 < a_n$ and $\sum_{n=1}^{\infty} a_n^2$ converges by the First Comparison Test.

8. Prove that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \log 2$.

Hint: Use $\sum_{k=0}^{n} x^k + \frac{x^{n+1}}{1 - x} = \frac{1}{1 - x}$ and integrate.

$$\int_0^{-1} \sum_{k=0}^{n} x^k \, dx + \int_0^{-1} \frac{x^{n+1}}{1 - x} \, dx = \int_0^{-1} \frac{dx}{1 - x}$$

or

$$\sum_{k=1}^{n+1} \frac{(-1)^k}{k} - \int_{-1}^{0} \frac{x^{n+1}}{1 - x} \, dx = \log 2 .$$

Hence

$$\left|\log 2 - \sum_{k=1}^{n+1} \frac{(-1)^k}{k}\right| = \left|\int_{-1}^{0} \frac{x^{n+1}}{1 - x} \, dx\right| \leq \int_{-1}^{0} \left|x^{n+1}\right| dx$$

$$\leq \int_0^1 x^{n+1} \, dx \leq \frac{1}{n + 2} .$$

9. (Cauchy Condensation Test) Show that if $n \to a_n$ is a decreasing sequence of positive terms then the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=0}^{\infty} 2^n a_{2^n}$ either both converge or both diverge.

The proof is based on the inequalities

$$2^n a_{2^n} = \sum_{k=2^n+1}^{2^{n+1}} a_{2^n} \geq \sum_{k=2^n+1}^{2^{n+1}} a_k$$

$$\geq \sum_{k=2^n+1}^{2^{n+1}} a_{2^{n+1}} = 2^n a_{2^{n+1}} = \frac{1}{2} 2^{n+1} a_{2^{n+1}} .$$

If $\sum_{k=1}^{\infty} a_k$ converges then

$$\sum_{k=1}^{n} 2^k a_{2^k} \leq 2 \sum_{k=1}^{2^n} a_k \leq 2 \sum_{k=1}^{\infty} a_k$$

and $\sum_{k=1}^{\infty} 2^k a_{2^k}$ converges since its sequence of partial sums is monotone and bounded.

If $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges, then for ν taken so that $2^\nu > n$, we have

$$\sum_{j=1}^{k} a_j \leq \sum_{j=1}^{2^\nu} a_j \leq a_i + \sum_{n=0}^{\nu-1} \left(\sum_{k=2^n+1}^{2^{n+1}} a_k \right)$$

$$\leq a_1 + \sum_{n=0}^{\nu-1} 2^n a_{2^n}$$

$$\leq a_1 + \sum_{n=0}^{\infty} 2^n a_{2^n} .$$

Thus the sequence of partial sums for $\sum_{j=1}^{k} a_j$ is both monotone and bounded; hence $\sum_{j=1}^{k} a_j$ converges.

10. (a) Use the Cauchy Condensation Test to show that $\sum_{k=2}^{\infty} \frac{1}{n \log n}$ diverges and that $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$ converges.

Since $\sum_{n=2}^{\infty} \frac{2^n}{2^n \log 2^n} = \sum_{n=2}^{\infty} \frac{1}{n \log 2}$ diverges, $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ diverges.

Since $\sum_{n=2}^{\infty} \frac{2^n}{2^n(\log 2^n)^2} = \sum_{n=2}^{\infty} \frac{1}{n^2(\log 2)^2}$ converges, $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$ converges.

(b) Apply the Cauchy Condensation Test to test the convergence of

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)(\log \log n)} .$$

Apply the Cauchy Condensation Test repeatedly to reduce the problem to convergence of

$$\sum_{n=1}^{\infty} \frac{2^n}{2^n(\log 2^n)(\log \log 2^n)} = \frac{1}{\log 2} \sum_{n=1}^{\infty} \frac{1}{n \log [n \log 2]}$$

and then to

$$\sum_{n=1}^{\infty} \frac{2^n}{2^n \log(2^n \log 2)} = \sum_{n=1}^{\infty} \frac{1}{n \log 2 + \log \log 2}$$

and then to

$$\sum_{n=1}^{\infty} \frac{2^n}{2^n \log 2 + \log \log 2} .$$

Since

$$\lim_{n\to\infty} \frac{2^n}{2^n \log 2 + \log\log 2} = \frac{1}{\log 2} \neq 0$$

it follows that the last series diverges and hence each of the other series in the succession diverges.

Solutions Exercises 14-4

1. Show that if $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ both converge, then $\sum_{n=1}^{\infty} a_n b_n$ converges.

   By the Cauchy inequality,

   $$\left(\sum_{k=1}^{n} |a_k b_k|\right)^2 \leq \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 \leq \sum_{k=1}^{\infty} a_k^2 \sum_{k=1}^{\infty} b_k^2 .$$

   Thus $n \to \sum_{k=1}^{n} |a_k b_k|$ is a monotone increasing sequence and therefore convergent. Since $\sum_{n=1}^{\infty} a_n b_n$ converges absolutely, it converges.

2. Test $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{(n+1)^2}$ for convergence.

   No. Since $\lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1 \neq 0$.

3. Is the following true in general? If $\sum_{n=1}^{\infty} a_n$ converges and $\lim_{n\to\infty} c_n = 0$, then $\sum_{n=1}^{\infty} a_n c_n$ converges.

It is not. Consider the following counterexample. Let $a_n = c_n = \frac{(-1)^n}{\sqrt{n}}$.

Then $\sum_{n=1}^{\infty} a_n$ converges by the Alternating Series Test, while

$\sum_{n=1}^{\infty} a_n c_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-test.

4. Test for convergence the alternating series $\sum_{k=1}^{\infty} a_k$, where $a_{2k+1} = \frac{1}{k}$ and $a_{2k} = -\frac{1}{2^k}$.

It diverges. If $\sum_{k=1}^{\infty} a_k$ converged then $\sum_{k=1}^{\infty} \frac{1}{k} = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} \frac{1}{2^k}$ would also converge.

5. Prove that if $|a_{n+1}a_{n-1}| < a_n^2$ for all n and $|a_2| < |a_1|$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

We have $\left|\frac{a_{n+1}}{a_n}\right| < \left|\frac{a_n}{a_{n-1}}\right|$; hence $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists by the Monotone Convergence Theorem. Moreover, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \le \left|\frac{a_2}{a_1}\right| < 1$. Hence $\sum_{n=1}^{\infty} |a_n|$ converges by the Ratio Test.

TC14-5. Parentheses and Rearrangements.

The striking result of Theorem 14-5a and Theorem 14-5b is that a very complicated condition is replaced by a simple one: If the limit of a convergent series can not be altered by rearrangement of its terms, then the series is absolutely convergent (and conversely).

In summary, the section demonstrates that convergence is sufficient for the extension of the associative law to infinite summation, but that the extension of the commutative law requires the stronger condition of absolute convergence.

Solutions Exercises 14-5

1. Prove Theorem 14-5b for the case $r < 0$.

   Choose i_1 so that $\sum_{k=1}^{i_1+1} a_k^- < r \le \sum_{k=1}^{i_1} a_k^-$. Choose $i_2 > i + 1$ so that

$$\sum_{k=1}^{i_1+1} a_k^- + \sum_{k=i_1+2}^{i_2} a_k^+ \le r < \sum_{k=1}^{i_1+1} a_k^- + \sum_{k=i_1+2}^{i_2} a_k^+ .$$

   Continuing in this manner construct a rearrangement which converges to r .

2. Show that $\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{y^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!}$ to conclude that $e^x e^y = e^{x+y}$. (See Section 8-5.)

By Theorem 14-5c (Cauchy Product),

$$\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{y^n}{n!}\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^k y^{n-k}}{k!(n-k)!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} x^k y^{n-k} \binom{n}{k}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} (x + y)^n .$$

Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ (Section 8-5), the conclusion follows.

TC14-6. Sequences of Functions. Uniform Convergence.

It is important to observe that the uniform convergence of $\sum_{n=1}^{\infty} u_n$ by itself is not sufficient to permit differentiation term-by-term; for this, uniform convergence of the series of derivatives, $\sum_{n=1}^{\infty} u'_n$, is the sufficient condition we employ (Theorem 14-6c). For example, the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ converges uniformly by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, but the series of derivatives $\sum_{n=1}^{\infty} \frac{\cos nx}{n}$ does not even converge at $x = 0$.

Solutions Exercises 14-6

1. Show that each of the following series converges uniformly on the sets specified.

(a) $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$, $-\infty < x < \infty$

$$\left|\frac{\sin nx}{n^2}\right| < \frac{1}{n^2}$$

(b) $\sum_{n=1}^{\infty} \frac{\sin x^n}{n}$, $|x| < \frac{1}{2}$

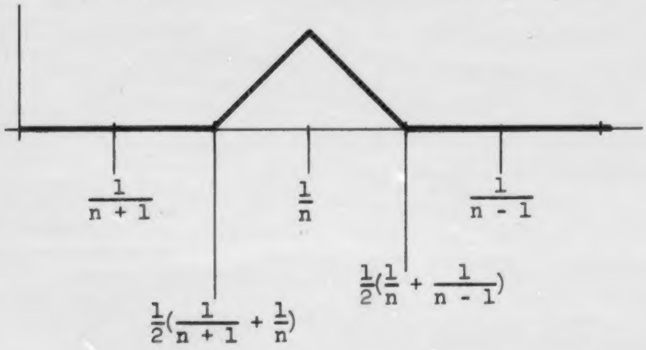
Since $|x| < \frac{1}{2} < \frac{\pi}{2}$, $|\sin x| \leq |x|$; hence, $\left|\frac{\sin x^n}{n}\right| \leq \frac{1}{n\,2^n}$.

(c) $\sum_{n=1}^{\infty} \left(\frac{x}{x+1}\right)^n$, $0 < x < 2$

For $0 < x < 2$, $(\frac{x}{x+1})^n < (\frac{2}{3})^n$.

2. Show that the Weierstrass M-Test is not a necessary condition for uniform convergence.

$$\text{Let } n \to U_n : x \to \begin{cases} 0 \text{ , for } 0 \le x \le \frac{1}{2}(\frac{1}{n} + \frac{1}{n+1}) \\ \qquad\qquad \text{and } \frac{1}{2}(\frac{1}{n} + \frac{1}{n-1}) \le x \le 1 \\ \frac{1}{n} \text{ , for } x = \frac{1}{n} \\ \text{linear between } \frac{1}{2}(\frac{1}{n+1} + \frac{1}{n}) \text{ and } \frac{1}{n} \\ \text{and between } \frac{1}{n} \text{ and } \frac{1}{2}(\frac{1}{n} + \frac{1}{n-1}) \end{cases} .$$



$\frac{1}{n+1}$
$\frac{1}{n}$
$\frac{1}{n-1}$
$\frac{1}{2}(\frac{1}{n} + \frac{1}{n-1})$
$\frac{1}{2}(\frac{1}{n+1} + \frac{1}{n})$

Then $\sum_{n=2}^{\infty} a_n(x)$ converges uniformly but $\max_{0 \le x \le 1} |a_n(x)| = \frac{1}{n}$ and $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges. Hence the Weierstrass M-Test cannot be applied to $\sum_{n=2}^{\infty} a_n(x)$.

3. Show that $\sum_{n=0}^{\infty} x^n$ does not converge uniformly on $|x| < 1$.

Take $\epsilon = \frac{1}{2}$ and $1 > x > \sqrt[n+1]{\frac{1}{2}}$.

Then $\left|\sum_{k=0}^{\infty} x^k - \sum_{k=1}^{n} x^k\right| = \frac{x^{n+1}}{1 - x} > x^{n+1} > \frac{1}{2}$.

Thus $\sum_{k=0}^{\infty} x^k$ does not converge uniformly on $|x| < 1$.

4. In Example 14-6a and Example 14-6b show for each fixed $\epsilon < 1$, $\Omega(x,\epsilon)$ **cannot** be bounded.

In both cases, for every $\epsilon < 1$ no matter what number ω is proposed as a bound for $\Omega(x,\epsilon)$, it is always possible to find x and $n > \omega$ such that $|u_n(x) - f(x)| > \epsilon$.

In Example 14-6a, we have for any $n > \omega$ and $0 < x < \frac{1 - \epsilon}{2^n}$,

$$|u_n(x) - f(x)| = u_n(x) < \epsilon \ .$$

In Example 14-6b, we have for any $n > \omega$ and $\frac{\epsilon}{4n^2} < x < \frac{1}{4n} - \frac{\epsilon}{4n^2}$,

$$|u_n(x) - f(x)| = u_n(x) > \epsilon \ .$$

5. A sequence of functions $u : n \to u_n$ is said to converge in the mean to f on $[a,b]$ if

$$\lim_{n \sim \infty} \int_a^b [f(x) - u_n(x)]^2 dx = 0 \ .$$

(a) Prove if u converges uniformly to f on $[a,b]$ then u converges in the mean to f .

Given $\epsilon > 0$, there exists ω such that if $n > \omega$ then

$$|f(x) - u_n(x)| < \sqrt{\frac{\epsilon}{b - a}} \ .$$

Thus

$$\left|\int_a^b (f(x) - u_n(x))^2 dx\right| = \int_a^b |f(x) - u_n(x)|^2 dx$$

$$< \int_a^b \frac{\epsilon}{b - a}\, dx = \epsilon \ .$$

(b) Show by an example that u can converge in the mean to f but not pointwise.

Take $u_n : x \to \begin{cases} 1 \text{ , for } 0 \le x \le \frac{1}{n} \text{ ,} \\ 0 \text{ , for } \frac{1}{n} \le x \le 1 \text{ ;} \end{cases}$

and $f : x \to 0$. Then

$$\lim_{n\sim\infty} \int_0^1 [f(x) - u_n(x)]^2 dx = \lim_{n\sim\infty} \frac{1}{n} = 0 \ ;$$

but

$$\lim_{n\sim\infty} u_n(0) = 1 \ne f(0) = 0 \ .$$

6. Show that if the series

(i) $$\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

converges uniformly to $f(x)$ on $[-\pi,\pi]$, then

(ii) $$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \ , \qquad n = 0 \ , 1 \ , \ldots$$

(iii) $$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \ , \qquad n = 1 \ , 2 \ , \ldots \ .$$

(The series (i) with coefficients defined by Equations (ii) and (iii) is called the Fourier Series of f .)

We have $\int_{-\pi}^{\pi} \cos nx \sin mx \, dx = 0$ for all m , n ; and

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \begin{cases} 0 \text{ , for } n \ne m \\ \pi \text{ , for } n = m = 1,2,\ldots \ . \end{cases}$$

Multiply by sin nx or cos nx in (i) and apply Theorem 14-6b to obtain

$$\int_{-\pi}^{\pi} \sin nx \, f(x) \, dx = \int_{-\pi}^{\pi} b_n \sin^2 nx \, dx = \pi b_n \ ,$$

$$\int_{-\pi}^{\pi} \cos nx \, f(x) \, dx = \int_{-\pi}^{\pi} a_n \cos^2 nx \, dx = \pi a_n \ ,$$

and

$$\int_{-\pi}^{\pi} f(x) dx = \int_{-\pi}^{\pi} \frac{a_0}{2} \, dx = \pi a_0 \ .$$

Solutions Exercises 14-7

1. If the series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ converge on $|x| < R$ show that

(i) $$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n$$

and that

(ii) $$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n$$

on any interval $|x| < p$ where $p < R$, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

(i) Use Theorem 14-3a.

(ii) Use Theorem 14-5c.

2. From Theorem 14-7a prove the claim of the text that a power series $\sum_{n=0}^{\infty} a_n x^n$ either

(a) converges for all x , or

(b) there exists a number R such that the series converges for $|x| < R$ and diverges for $|x| > R$.

If the set of values, E , for which the series converges is unbounded, then $E = R$ by Theorem 14-7a, since, given any x there exists y , $|y| > x$, and the series converges at y . If E is bounded let $R = \sup E$. Thus, if $|x| > R$ the series must diverge. If $|x| < R$ there is a y in E with $|x| < y$. Hence by Theorem 14-7a, the series converges at x .

3. Prove that if $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R_1 and $\sum_{n=0}^{\infty} b_n x^n$ has radius of convergence $R_2 < R_1$, then $\sum_{n=0}^{\infty} (a_n + b_n) x^n$ has radius of convergence R_2 .

It is clear that $\sum_{n=0}^{\infty} (a_n + b_n)x^n$ converges for $|x| < R_2$. If $\sum_{n=0}^{\infty} (a_n + b_n)x^n$ converges where $R_2 < x < R_1$ then since

$$\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} (a_n + b_n)x^n - \sum_{k=0}^{\infty} b_n x^n$$

the difference

$$\sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n)x^n - \sum_{n=0}^{\infty} a_n x^n$$

converges, but this is impossible since x lies outside the interval of convergence of $\sum_{n=0}^{\infty} b_n x^n$.

4. Show that the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ is

$$R = \begin{cases} 0 & \text{if } \overline{\lim} \sqrt[n]{|a_n|} = \infty \\ \infty & \text{if } \overline{\lim} \sqrt[n]{|a_n|} = 0 , \end{cases}$$

in all other cases $R = \overline{\lim} \dfrac{1}{\sqrt[n]{|a_n|}}$.

Choose r_1 so that $r < r_1 < R$. Then $\overline{\lim} \sqrt[n]{|a_n|} = \frac{1}{R} < \frac{1}{r_1} < \frac{1}{r}$. Hence, there exists ω such that for $k > \omega$, $\sqrt[k]{|a_k|} < \frac{1}{r_1}$. Since $|x| < r$ we have $\sqrt[k]{|a_k|}\, |x| < \frac{r}{r_1}$ or $|a_k||x|^k < (\frac{r}{r_1})^k$ for $k > \omega$. Since $\frac{r}{r_1} < 1$, the comparison series $\sum_{k=0}^{\infty} (\frac{r}{r_1})^k$ converges and $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly in $|x| < r$ by the Weierstrass M-Test.

If $|x| > R$, choose r_2 so that $|x| > r_2 > R$. Then $\frac{1}{r_2} < \frac{1}{R} = a$ so then in a subsequence $\sqrt[n_j]{|a_{n_j}|}$ satisfying $\sqrt[n_j]{|a_{n_j}|} > \frac{1}{r_2}$ (Exercises 14-2, No. 20). Hence $\sqrt[n_j]{|a_{n_j}|}\,|x| > \frac{|x|}{r_2}$ or $|a_{n_j}|\,|x|^{n_j} > (\frac{|x|}{r_2})^{n_j} > 1$. Thus $\sum_{n=0}^{\infty} a_n x^n$ diverges by the n-th term test.

5. Find the radius of convergence, R, for each of the following power series

(a) $\sum_{n=0}^{\infty} n(n + 1)x^n$;

(b) $\sum_{n=1}^{\infty} \frac{2^n x^n}{n}$;

(c) $\sum_{n=0}^{\infty} \frac{n^k x^n}{n!}$;

(d) $\sum_{n=0}^{\infty} \frac{n! x^n}{n^n}$.

Apply the Ratio Test in (a), (b), and (c) to obtain

(a) $R = 1$, (b) $R = \frac{1}{2}$, and (c) $R = \infty$.

In (d), by Section 8-6 Equation (13), $2(\frac{n}{e})^n \leq n! \leq 4n(\frac{n}{e})^n$ so that $\frac{|x|}{e}\sqrt[n]{2} \leq \sqrt[n]{n!(\frac{x}{n})^n} \leq \frac{|x|}{e}\sqrt[n]{4n}$ whence $\lim_{n\to\infty} \sqrt[n]{\frac{n!|x|^n}{n}} = \frac{|x|}{e}$ by the Squeeze Theorem. Hence $R = e$.

Alternatively, apply the Ratio Test to obtain

$$R = \lim_{n\to\infty} (1 + \frac{1}{n})^n = e .$$

Solutions Miscellaneous Exercises

1. Extend the Second Comparison Test by proving that if $\lim\limits_{n\sim\infty} \frac{b_n}{a_n} = 0$, where $a_n > 0$, for all n, and the $\sum\limits_{n=1}^{\infty} a_n$ converges then $\sum\limits_{n=1}^{\infty} b_n$ converges absolutely.

   There exists an ω such that for $k > \omega$,

$$\left|\frac{b_k}{a_k}\right| < 1 ,$$

   or $|b_k| < a_k$. Hence by the First Comparison Test $\sum\limits_{n=1}^{\infty} |b_n|$ converges.

2. Let $\sum\limits_{n=1}^{\infty} b_n$ be a convergent series of positive terms. Prove if $\left|\frac{a_{n+1}}{a_n}\right| < \frac{b_{n+1}}{b_n}$, for all n, then $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely.

   The sequence $n \to \frac{|a_n|}{b_n}$ is decreasing and bounded below by 0, hence convergent. By the Second Comparison Test, or by the preceding exercise, if $\lim\limits_{n\sim\infty} \frac{|a_n|}{b_n} = 0$, the convergence of $\sum\limits_{n=1}^{\infty} b_n$ implies the convergence of $\sum\limits_{n=1}^{\infty} |a_n|$.

3. Let $\sum\limits_{n=1}^{\infty} a_n$ be a series of nonnegative terms. Prove if $n\left(\frac{a_n}{a_{n+1}} - 1\right) > 1 + \epsilon > 1$ for $n > \omega$, then $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely, but if $n\left(\frac{a_n}{a_{n+1}} - 1\right) < 1 - \epsilon < 1$ then $\sum\limits_{n=1}^{\infty} a_n$ diverges.

Hint: Use the preceding exercise to compare the given series with a p-series, where $p = 1 \pm \epsilon$.

If $n\left(\frac{a_n}{a_{n+1}} - 1\right) > 1 + \epsilon$ for $n > \omega$, then $\frac{a_{n+1}}{a_n} < \frac{1}{1 + \frac{1+\epsilon}{n}} < \frac{1}{(1 + \frac{1}{n})^{1+\epsilon}}$.

Apply Number 2,

$$\frac{\frac{1}{(n+1)^{1+\epsilon}}}{\frac{1}{n^{1+\epsilon}}} = \left(\frac{n}{n+1}\right)^{1+\epsilon} = \frac{1}{(1 + \frac{1}{n})^{1+\epsilon}} .$$

Thus the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}}$ implies the convergence of

$\sum_{n=1}^{\infty} |a_n|$. Divergence is handled analogously.

4. Show that each of the conditions in Leibniz's Test (Theorem 14-4b) is necessary for convergence.

The divergent series $\sum_{n=1}^{\infty} \frac{1}{n}$ satisfies conditions 2 and 3 of Theorem 14-4b, but not condition 1. The divergent series of Exercises 14-4, Number 4 satisfies conditions 1 and 3, but not condition 2. The divergent series $\sum_{n=1}^{\infty} (-1)^n(1 + \frac{1}{n})$ satisfies conditions 1 and 2, but not condition 3.

5. Prove that $\sum_{i=1}^{\infty} a_i$ converges if

(a) $\operatorname{sgn} a_k = -\operatorname{sgn} a_{k+1}$

(b) $k \to |a_k|$ is nonincreasing

(c) $\lim_{k \to \infty} a_k = 0$.

As in Theorem 14-4b we find that

$$s_e : n \to s_{2n} = \sum_{k=1}^{2n} a_k$$

and

$$s_0 : n \to s_{2n+1} = \sum_{k=1}^{2n+1} a_k$$

are monotone sequences (one is nonincreasing and the other nondecreasing rather than increasing and decreasing as in the proof of Theorem 14-4b). Then continue as in Theorem 14-4b.

6. Show that if $\lim_{n\sim\infty} \left|\frac{a_{n+1} - a_n}{a_n - a_{n-1}}\right| = r$, then the sequence $n \to a_n$ converges if $r < 1$ and diverges if $r > 1$.

Define the sequence $b : k \to b_k = a_k - a_{k-1}$. Then $a_k = a_0 + \sum_{k=1}^{n} b_k$ and we are given $\lim_{k\sim\infty} \left|\frac{b_{k+1}}{b_k}\right| = r$; hence $n \to a_n$ converges if $r < 1$ and diverges if $r > 1$.

7. Translate the Weierstrass M-Test as a criterion for the uniform convergence of a sequence of function $u : n \to u_n$.

u_n is the n-th partial sum of the series $\sum_{n=1}^{\infty} (u_k - u_{k-1})$, where we set $u_1 = 0$. Thus the sequence u converges uniformly if there exists a convergent series of constants $\sum_{n=1}^{\infty} M_n$ with $|u_n - u_{n-1}| \leq M_n$.

8. If, for all x in E, $|u_\nu(x)| \leq M$ and $\left|\frac{u_{n+1}(x)}{u_n(x)}\right| < r < 1$ for all $n \geq \nu$, then $\sum_{n=1}^{\infty} u_n$ converges uniformly in E.

Compare $\sum_{n=1}^{\infty} u_n$ with a geometric series. For every integer $k > 0$ we have

$$|u_{\nu+k}(x)| \leq r|u_{\nu+k-1}(x)| \leq \cdots$$
$$\leq r^k |u_\nu(x)| \leq r^k M$$

for all x in E. Hence $\sum_{n=1}^{\infty} u_n$ converges uniformly in E by the Weierstrass M-Test.

9. A telescoping series is a series of the form $\sum_{n=1}^{\infty} (a_n - a_{n+1})$. Give necessary and sufficient conditions for the convergence of a telescoping series.

The n-th partial sum of the telescoping series $\sum_{n=1}^{\infty} (a_n - a_{n+1})$ is $s_n = \sum_{k=1}^{n} (a_k - a_{k+1}) = a_1 - a_{n+1}$. Thus the convergence of $\sum_{k=1}^{\infty} (a_k - a_{k+1})$ is equivalent to the convergence of $n \to a_n$. If $\lim_{n\to\infty} a_n = r$, then $\sum_{n=1}^{\infty} a_n = a_1 - r$.

10. Prove the Cauchy Criterion for uniform convergence: a necessary and sufficient condition for the uniform convergence of the sequence of functions $u : n \to u_n$ with common domain E is that to every $\epsilon > 0$ there exists an $\omega = \Omega(\epsilon)$ such that if $n, m > \omega$, then $|u_n(x) - u_m(x)| < \epsilon$ for all x in E.

Suppose that the sequence u has the property stated. Then for each x in E the sequence $n \to u_n(x)$ converges by the Cauchy Convergence Theorem, say to $U(x)$; i.e., u converges pointwise to $U : x \to U(x)$. Given $\epsilon > 0$, if $n > \omega_1 = \Omega(\frac{\epsilon}{2})$, then for any $k > \omega_1$,

$$\begin{aligned} |U(x) - u_n(x)| &\leq |U(x) - u_k(x)| + |u_k(x) - u_n(x)| \\ &< |U(x) - u_k(x)| + \frac{\epsilon}{2} . \end{aligned}$$

Now for each x take any k large enough so that $|U(x) - u_k(x)| < \frac{\epsilon}{2}$. Thus, $|U(x) - u_n(x)| < \epsilon$ for all x in E if $n > \omega_1$. (We have obtained an ω independent of x , although the details of the proof involve a choice of k which does not depend on x .) Conversely, if u converges uniformly in E , say to U , to each $\epsilon > 0$ we can associate $\omega = \Omega(\epsilon)$ so that for $n > \omega$, $|U(x) - u_n(x)| < \epsilon$ for all x in E . Hence if m , $n > \omega_1$ where $\omega_1 = \Omega(\frac{\epsilon}{2})$ we have, for all x in E ,

$$\begin{aligned} |u_n(x) - u_m(x)| &\leq |U(x) - u_n(x)| + |U(x) - u_m(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq \epsilon . \end{aligned}$$

11. Show if the series of functions $\sum_{n=1}^{\infty} v_n$ converges uniformly on E and the series of functions $\sum_{n=1}^{\infty} u_n$, with common domain E , has the property that $|u_n(x)| \leq v_n(x)$ for all $x \in E$, then $\sum_{n=1}^{\infty} u_n$ converges uniformly in E .

Given any $\epsilon > 0$, there exists an ω such that, if $n > m > \omega$, then

$$\sum_{k=m+1}^{n} v_k(x) = \left| \sum_{k=m+1}^{n} v_k(x) \right| < \epsilon$$

for all $x \in E$. Consequently,

$$\left| \sum_{k=m+1}^{n} u_k(x) \right| \leq \sum_{k=m+1}^{n} |u_k(x)| \leq \sum_{k=m+1}^{n} v_k < \epsilon$$

for all $x \in E$ and $n > m > \omega$. Hence, $\sum_{n=1}^{\infty} u_n$ converges uniformly in E by the Cauchy-Criterion for uniform convergence. (See Exercises 14-M, No. 10.)

12. (a) Consider a series of functions $\sum_{n=1}^{\infty} u_n$ uniformly convergent to U on E. Let f be a function defined and bounded on E, $|f(x)| \leq M$. Prove that $\sum_{n=1}^{\infty} f \cdot u_n$ converges uniformly to $f \cdot \sum_{n=1}^{\infty} u_n = U$.

For each positive ϵ, there exists an $\omega = \Omega(\epsilon)$ such that $\left|U(x) - \sum_{k=1}^{n} u(x)\right| < \epsilon$ whenever $n > \omega$. If $n > \omega_1 = \Omega(\frac{\epsilon}{M})$, then

$$|f(x)U(x) - \sum_{k=1}^{n} f(x)u_k(x)| = |f(x)||U(x) - \sum_{k=1}^{n} u(x)|$$

$$< M \cdot \frac{\epsilon}{M} \leq \epsilon .$$

(b) Show, by an example, that the boundedness of f is a necessary condition in Part (a).

Let $E = (0,1)$, $u_n : x \to (\frac{1}{2})^n$ and $f : x \to \frac{1}{x}$. Then $\sum_{k=0}^{\infty} \frac{1}{x} \cdot \frac{1}{2^k} = \frac{1}{x}$. But

$$\left|\frac{1}{x} - \sum_{k=1}^{n} \frac{1}{x\ 2^k}\right| = \frac{1}{x} \cdot \frac{1}{2^n}$$

$$> 1 \text{ for } x < \frac{1}{2^n} .$$

13. Find the Taylor expansion of $f : x \to (1 + x)^{\alpha}$, α not a positive integer, (the binomial series for exponent α), and find its radius of convergence.

By induction, show that $f^{(k)}(x) = \alpha(\alpha - 1) \ldots (\alpha - k + 1)x^{\alpha-k}$. Thus, the binomial series with exponent α is $\sum_{k=0}^{\infty} \frac{\alpha(\alpha - 1) \ldots (\alpha - k + 1)x^k}{k!}$.

Applying the Ratio Test,

$$\lim_{k\sim\infty} \left|\frac{\alpha(\alpha - 1) \ldots (\alpha - k)x^{k+1}}{(k + 1)!} \cdot \frac{k!}{\alpha(\alpha - 1) \ldots (\alpha - k + 1)x^k}\right|$$

$$= \lim_{k\sim\infty} \left|\frac{\alpha - k}{k + 1} x\right| = |x| ,$$

we see that the series converges for $|x| < 1$ and diverges for $|x| > 1$. Hence 1 is the radius of convergence of the binomial series.

By using the Cauchy form of the remainder in Taylor's Theorem, (Exercises 13-3, No. 9(b)), it is possible to prove that the series not only converges, but converges to the function f for $|x| < 1$. The technique of proof parallels that of Exercises 13-3, Number 9(c).

14. Show that the radius of convergence, R , of the Taylor series of arc sin x ,

$$\sum_{k=0}^{\infty} \frac{(2k)!t^{2k+1}}{(2k + 1)(k!)^2\, 2^{2k}} ,$$

(see Example 13-3b) is 1 .

Solution 1: By the Ratio Test, since

$$\lim_{k\sim\infty} \left|\frac{t^{2k+3}(2k + 2)!}{(2k + 3)[(k + 1)!]^2\, 2^{2k+2}} \cdot \frac{(2k + 1)(k!)^2\, 2^{2k}}{(2k)!\, t^{2k+1}}\right| = t^2 ,$$

the series converges for $|t^2| < 1$ and diverges for $|t| > 1$, whence the radius of convergence is 1 .

Solution 2: By Exercises 14-7, Number 4,

$$\frac{1}{R} = \overline{\lim} \sqrt[2k+1]{\frac{(2k)!}{(2k + 1)(k!)^2\, 2^{2k}}} .$$

Now apply Stirling's formula (Section 13-4) to obtain,

$$\frac{1}{R} = \overline{\lim} \sqrt[2k+1]{\frac{\sqrt{4\pi k}\,(\frac{2k}{e})^{2k}}{(2k+1)[\sqrt{2k\pi}(\frac{k}{e})^{k}]^2\, 2^{2k}}}$$

$$= \overline{\lim} \sqrt[2k+1]{\frac{\sqrt{4\pi k}}{(2k+1)2\pi k}}$$

$$= 1 .$$

15. Show that if the continuous function $(x,y) \to \Phi(x,y)$ defined in the rectangle

$$\{(x,y) : |x - x_0| \le a , |y - y_0| \le c\}$$

satisfies $ab < 1$, $a\Lambda \le c$, where

(1) $\max\{|\Phi(x,y)| : |x - x_0| \le a , |y - y_0| \le c\} = \Lambda$

and

(2) $\max\{|D_y \Phi(x,y)| : |x - x_0| \le a , |y - y_0| \le c\} = b$,

then the sequence of functions $u : k \to u_k$ defined by $u_0 : x \to y_0$,

$$u_{k+1}(x) = \int_{x_0}^{x} \Phi(x,u_k(x))dx$$

converges to a function U which satisfies the differential equation

$$\frac{dy}{dx} = \Phi(x,y) \quad \text{for} \quad |x - x_0| < a ,$$

and the initial condition $y = y_0$ at $x = x_0$.

$$|u_0(x) - y_0| \le c \quad \text{for} \quad |x - x_0| \le a .$$

Proceed by induction:

if $|u_k(x) - y_0| \le c$ for $|x - x_0| \le a$, then

$$|u_{k+1}(x) - y_0| \le \int_{x_0}^{x} |\Phi(x , u_k(x))|dx \le a\Lambda \le c .$$

Now, let $\lambda_k = \max\{|u_k(x) - u_{k-1}(x)| : |x - x_0| \le a\}$. Then, for $|x - x_0| \le a$,

$$|u_{k+1}(x) - u_k(x)| \le \int_{x_0}^{x} |\Phi(x , u_k(x) - \Phi(x , u_{k-1}(x))|dx .$$

The integral in this inequality is equal to

$$\int_{x_0}^{x} |D_y \Phi(x,y)||u_k(x) - u_{k-1}(x)|dx$$

for some $y = f(x)$ between $u_{k-1}(x)$ and $u_k(x)$ Hence,

$$|u_{k+1}(x) - u_k(x)| < \int_{x_0}^{x} b \cdot \lambda_k \, dx \leq ab\,\lambda_k ,$$

or $\lambda_{k+1} \leq ab\lambda_k$. By induction, $\lambda_{k+1} \leq (ab)^k\lambda_1$. Since, for $|x - x_0| \leq a$,

$$\begin{aligned}
|u_{n+k}(x) - u_n(x)| &= \left|\sum_{j=1}^{k} u_{n+j}(x) - u_{n+j-1}(x)\right| \\
&\leq \sum_{j=1}^{k} |u_{n+j}(x) - u_{n+j-1}(x)| \\
&\leq \sum_{j=1}^{k} \lambda_{n+j} \\
&\leq \lambda_1 \sum_{j=1}^{k} (ab)^{n+j-1} \\
&\leq \lambda_1 (ab)^n \sum_{j=0}^{\infty} (ab)^j \\
&\leq \frac{\lambda_1 (ab)^n}{1 - ab} .
\end{aligned}$$

Since the sequence $n \rightarrow (ab)^n$ converges to 0 , the sequence u satisfies the Cauchy Criterion for uniform convergence. Consequently, u converges uniformly to a function U . For each fixed x ,

$$n \rightarrow \Phi(x,u_n(x)) = u'_{n+1}(x)$$

converges to $\Phi(x,U(x))$ by Theorem 14-2g. In fact, the convergence is uniform since

$$\begin{aligned}
|\Phi(x,U(x)) - \Phi(x,u_n(x))| &= |D_y\Phi(x,y)||U(x) - u_n(x)| \\
&\leq b \max\{|U(x) - u_n(x)| : |x - x_0| \leq a\} .
\end{aligned}$$

Thus, by Theorem 14-6d $\frac{dU}{dx} = \Phi(x,U)$ for $|x - x_0| \leq a$ and $U(x_0) = y_0$.

AA16. Find the value of $a^{a^{a^{\cdot^{\cdot^{\cdot}}}}}$. More precisely, find the limit of the sequence $n \to x_n$ defined by $x_0 = a$, $x_{n+1} = a^{x_n}$ and determine the values of a for which the sequence converges.

If the sequence converges, say to r , then, by the elementary limit theorems, r satisfies $a^r = r$. Hence, if the sequence is to converge at all, a must be a value of the function $f : x \to x^{1/x}$ (see Figure (a)).

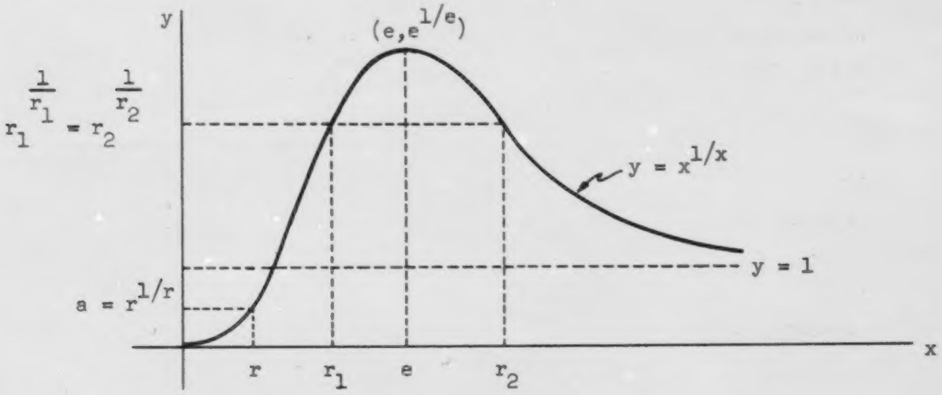


Figure (a)

The function f has a unique maximum, $e^{1/e}$, at e . Thus, for convergence of the sequence, we require $a \leq e^{1/e}$, and for $a > e^{1/e}$ the sequence must diverge.

First we shall show if $1 < r \leq e$, (hence, for $a = r^{1/r}$, $1 < a < r$) then $\lim_{n\sim\infty} x_n = r$, and if $a > e^{1/e}$ then $n \to x_n$ diverges. We proceed by the methods of Section 13-2. For the function $\phi : t \to (r^{1/r})^t = a^t$ we have

(i)
$$r - x_{n+1} = \phi(r) - \phi(x_n)$$
$$= \phi'(u)(r - x_n)$$

where u lies between x_n and r . Now when $1 < u < r$, we have the inequality for $\phi'(u) = a^u \log a$,

(ii)
$$0 < \phi'(u) < a^r \log a \leq r \log r^{1/r} \leq \log r \leq 1 .$$

Since $x_0 = a$, it follows that $0 < \phi'(a) < 1$ from (ii), hence from (i), that $r > x_1 > x_0$. Proceding by induction, we obtain

$$x_n < x_{n+1} < r .$$

Thus $n \to x_n$ is a bounded increasing sequence and converges. That $\lim_{n\sim\infty} x_n = r$ follows from the elementary limit theorems.

Next suppose $0 < r < 1$. In this case, with $a = r^{1/r}$, we have

(iii) $$0 < a < r < 1 .$$

Observe, for $0 < \theta < 1$ and $p < q$, that $\theta^p > \theta^q$; i.e., that the function $t : \theta^t$ is decreasing. With this observation, taking $\theta = a = r^{1/r}$ we have from (iii),

(iv) $$x_0 = 1 > x_2 > r > x_1 > 0$$

where we recall that $x_2 = a^a$ and $x_1 = a$. Again, taking $\theta = a$, we have from (iv)

$$x_1 < x_3 < r < x_2 < x_0 .$$

In general, if

$$r < x_{2k} < x_{2k-2}$$

we obtain, similarly,

$$r > x_{2k+1} > x_{2k-1} ,$$

and, hence,

$$r < x_{2k+2} < x_{2k} .$$

We see, then, that the sequences $k \to x_{2k-2}$ and $k \to x_{2k-1}$ are, respectively, decreasing and bounded below by r and increasing and bounded above by r. Thus, both sequences converge but they need not converge to the same limit and further investigation is necessary.

Set $p = \lim_{k\sim\infty} x_{2k}$ and $q = \lim_{k\sim\infty} x_{2k-1}$. Then

(v) $$p \geq r \geq q .$$

Now, from

$$a^{x_{2k-1}} = x_{2k} , \quad a^{x_{2k}} = x_{2k+1}$$

we have by the elementary limit theorems

(vi) $$a^q = p \, , \; a^p = q \; .$$

From (vi) we have also

$$a^{a^p} = p \, , \; a^{a^q} = q \; .$$

Thus p and q are both zeros of

(vii) $$\psi : t \to a^{a^t} - t \; .$$

Let us investigate the solutions of this equation. Observe first that r is a solution. To investigate the behavior of ψ we examine the behavior of the derivative ψ' where, with $\xi = a^{a^t}$,

$$\begin{aligned}\psi'(t) &= a^{a^t} a^t (\log a)^2 - 1 \\ &= \xi \log \xi \log a - 1 \; .\end{aligned}$$

In Figure (b), we sketch the graph

$$\eta = f(\xi) = \xi \log \xi \log a - 1 \quad \text{for} \quad 0 < \xi < 1 \; .$$

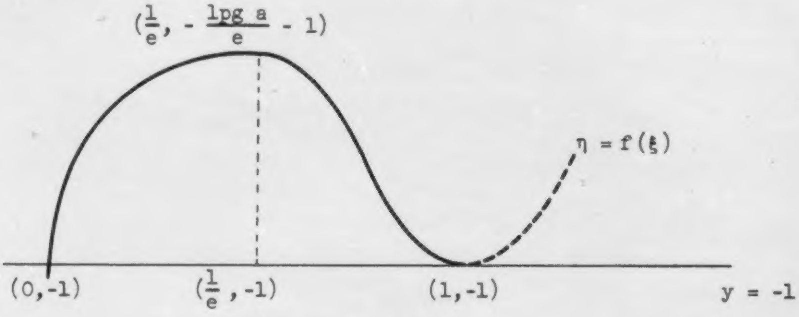


Figure (b)

In the interval $(0,1)$, f has a maximum, $\eta = -\frac{\log a}{e} - 1$ at $\xi = \frac{1}{e}$. If $a \geq (\frac{1}{e})^e$ then $\eta < 0$; hence, $\psi'(t) = f(\xi) < 0$ for $0 < x < 1$ (except for the limiting case $a = (\frac{1}{e})^e$ when there is an isolated zero at $\xi = \frac{1}{e}$, but this does not affect the monotone character of ψ; see Exercises 5-4, No. 14). Consequently, for $a \geq (\frac{1}{e})^e$, the zero of (vii) is unique. We conclude in this case that $p = q = r$ and the sequence converges (Exercises 14-2, No. 15).

Finally, we prove divergence for $r < \frac{1}{e}$, or equivalently, $0 < a < (\frac{1}{e})^e$. Recall that we have strict inequality

$$x_{2k-1} < r < x_{2k}$$

for all k so that $x_n \neq r$ for any term of the sequence. Now since $r < \frac{1}{e}$ we have for $\phi : r \to a^r$,

$$\phi'(r) = a^r \log a = \log r < -1 .$$

Since ϕ' is continuous, it follows that $\phi'(t) = a^t \log a < -C < -1$ on some neighborhood I of $t = r$. (Here we may fix $C = \frac{1 - \log r}{2} > 1$ or use any constant between -1 and $\log r$.) Now we show that for every ω there is an index k for which x_k lies outside I, hence the sequence cannot converge to r and therefore diverges. For suppose $x_i \in I$ for some x_i, where $i > \omega$. If x_{i+1} is not in I, we are done. If $x_{i+1} \in I$, then from

$$\begin{aligned} x_{i+1} - r &= \phi(x_i) - r \\ &= \phi(x_i) - \phi(r) \\ &= \phi'(u_i)(x_i - r) \end{aligned}$$

where u_i is between x_i and r. Note that

(viii) $$\begin{aligned} |x_{i+1} - r| &= |\phi'(u_i)||x_i - r| \\ &> C|x_i - r| ; \end{aligned}$$

thus x_{i+1} is farther from r than x_i. Since $x_{i+1} \in I$, we have on replacing i by $i + 1$ in (viii),

$$|x_{i+2} - r| > C|x_{i+1} - r|$$

and from (viii) we conclude that

$$|x_{i+2} - r| > C^2|x_i - r| .$$

Now x_{i+2} is outside I, or we may apply (viii) in the same way and obtain

$$|x_{i+3} - r| > C^3|x_i - r| .$$

The process must terminate since $C > 1$, for $C^j|x_i - r|$ is greater than the radius of I for some sufficiently large j, and if no term between x_i and x_{i+j} lies outside I, then

$$|X_{i+j} - r| > C\,|X_i - r|$$

and X_{i+j} must lie outside I. Thus the proof of divergence is complete.

Teacher's Commentary

Chapter 15

GEOMETRICAL OPTICS AND WAVES

In this chapter we have attempted to show how broadly the methods of the calculus enter into the development of a science (in contrast to Chapters 9 and 12 which pursued restricted paths of mathematical development). The science of optics chosen for this purpose is exceedingly rich and leads naturally to mathematical ideas outside the confines of the text (in particular, the ideas of multivariate calculus). This would remain true even if it were possible to extend the text beyond its present frame and include traditional multivariate calculus and much of higher analysis. That being so, the chapter comes fitly at the end as both a remembrance of things past and a forerunner of things to come.

The text of the chapter is designed rather for individual reading than group classroom activity. The approach is mathematically informal and nonrigorous as is appropriate for a cursory exploration of a broad area; such matters as exact error analysis can wait until the student develops a more specialized interest in the subject.

Solutions Exercises 15-2

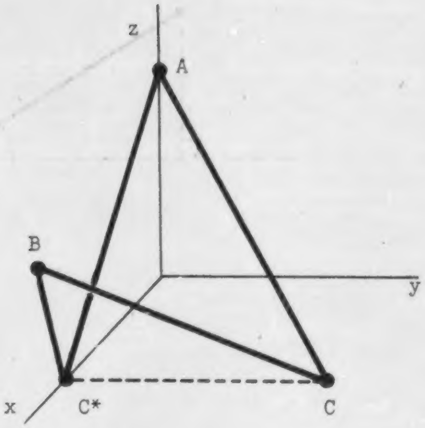
1. Show that the shortest path from a point A to a point B by way of a point on a plane mirror, must necessarily lie in the plane containing A and B which is perpendicular to the plane of the mirror.

Choose x, y, z-coordinates so that the plane of the mirror is given by $z = 0$ and the plane through A and B perpendicular to the mirror plane is given by $x = 0$. Locate the origin so that A is on the z-axis. Thus $A = (0,0,a_3)$ and $B = (b_1,0,b_3)$. Let $C = (c_1,c_2,0)$ be any point on the mirror plane. Next, show that if C is not in the plane $x = 0$, $c_2 \neq 0$, then the path ACB is longer than the path AC^*B where $C^* = (c_1,0,0)$, (see figure), as follows. The two path lengths are

$$L = |\overline{AC}| + |\overline{CB}|$$

and

$$L^* = |\overline{AC^*}| + |\overline{C^*B}| .$$



But, from

$$|\overline{AC}| = \sqrt{a_3^2 + c_1^2 + c_2^2}$$

and

$$|\overline{AC^*}| = \sqrt{a_3^2 + c_1^2} ,$$

it is immediate that $|\overline{AC}| \geq |\overline{AC^*}|$. Similarly,

$$|\overline{C^*B}| = \sqrt{(c_1 - b_1)^2} \leq \sqrt{(c_1 - b_1)^2 + c_2^2 + b_3^2} = |\overline{CB}| .$$

On addition, $L > L^*$ follows.

2. (a) Equation (2) is a necessary but not sufficient condition for the path length L to be a minimum. Show, in fact, that the condition $\alpha = \gamma$ is sufficient for a minimum.

Differentiate again in

$$\frac{dL}{dx} = \frac{x}{L_1} - \frac{d - x}{L_2} = \sin \alpha - \sin \gamma$$

to obtain

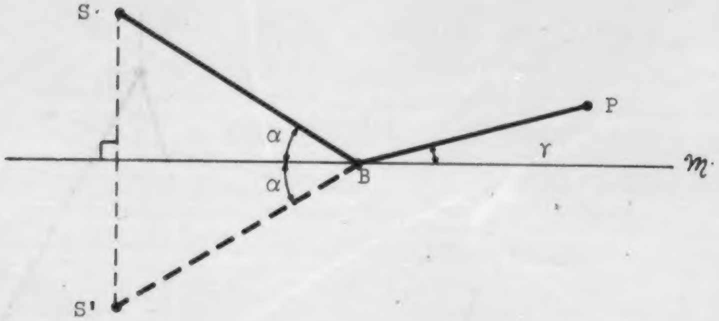
$$\frac{d^2L}{dx^2} = \frac{1}{L_1} - \frac{x}{L_1^2}\frac{dL_1}{dx} + \frac{1}{L_2} + \frac{d - x}{L_2^2}\frac{dL_2}{dx}$$

$$= \frac{1}{L_1}(1 - \sin^2 \alpha) + \frac{1}{L_2}(1 - \sin^2 \gamma)$$

$$= \frac{1}{L_1}\cos^2 \alpha + \frac{1}{L_2}\cos^2 \gamma > 0 .$$

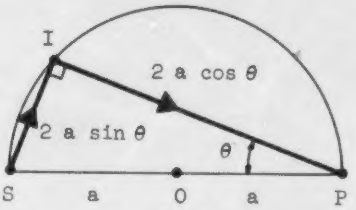
Thus the graph of L is flexed upward and $\frac{dL}{dx} = 0$ is a sufficient condition for an absolute minimum.

(b) Show that $\gamma = \alpha$ corresponds to the shortest path by the methods of elementary geometry. (Hint: Use the image principle. This was the method used originally by Hero.)



Let S' by the image of S in the mirror plane $\mathcal{M}$. Let B be any point of . The paths SBP and S'BP are equal in length, but S'BP will be minimal if and only if S'BP is straight. In that case $\gamma = \alpha$ as claimed.

3. Show that [E2] yields the longest possible reflection path between diametrically opposite points of a circular reflector (Figure 15-2f(i)).



From the accompanying figure, the path length is given as

$$L = 2a(\cos\theta + \sin\theta) ,$$

where $0 < \theta < \frac{\pi}{2}$. Consequently, a zero of

$$\frac{dL}{d\theta} = 2a(-\sin\theta + \cos\theta)$$

can occur in the parameter interval only at $\theta = \frac{\pi}{4}$ as predicted by [E2] . But

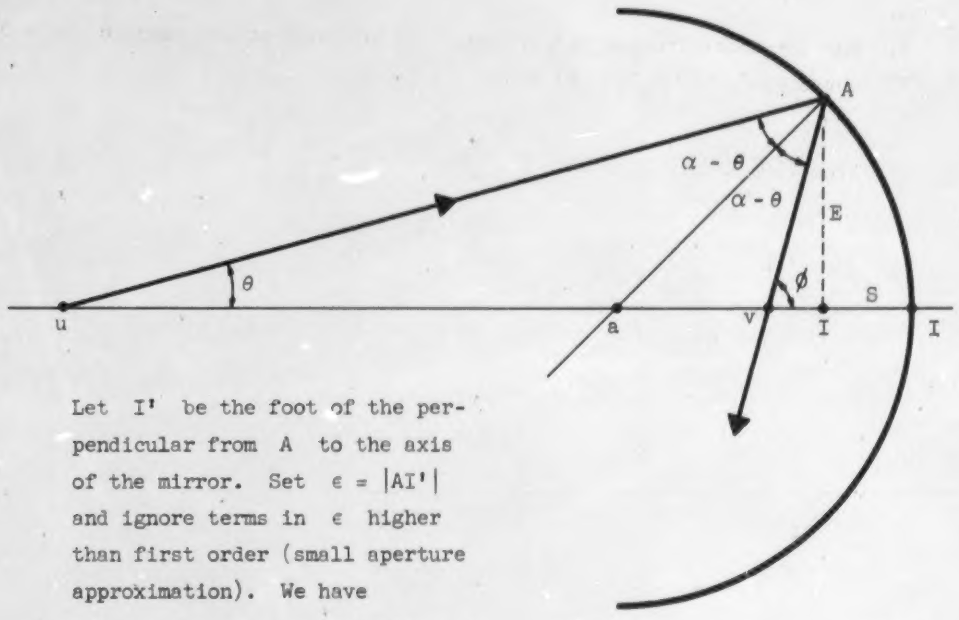
$$\left.\frac{dL}{d\theta}\right|_{\theta=0} > 0 \quad \text{and} \quad \left.\frac{dL}{d\theta}\right|_{\theta=\frac{\pi}{2}} < 0 .$$

We conclude that L has a maximum at $\frac{\pi}{4}$.

4. Show to a first approximation for a small aperture concave mirror (Figure 15-2f(ii)) that all rays from a source on the axis of the mirror at distance u from the mirror center are reflected through a point at distance v from the mirror, where

$$\frac{1}{u} + \frac{1}{v} = \frac{2}{a} .$$

15-2



Let I' be the foot of the perpendicular from A to the axis of the mirror. Set $\epsilon = |AI'|$ and ignore terms in ϵ higher than first order (small aperture approximation). We have

$$\delta = a - \sqrt{a^2 - \epsilon^2} \approx a\left(1 - \sqrt{1 - \frac{\epsilon^2}{a^2}}\right) \approx \frac{\epsilon^2}{2a} \approx 0 \ ,$$

where a is the radius of the mirror. Consequently,

$$\tan \phi = \frac{\epsilon}{v - \delta} \approx \frac{\epsilon}{v} \ ,$$

$$\tan \alpha = \frac{\epsilon}{a - \delta} \approx \frac{\epsilon}{a} \ ,$$

$$\tan \theta = \frac{\epsilon}{u - \delta} \approx \frac{\epsilon}{u} \ ,$$

and

$$\tan \phi = \tan(2\alpha - \theta) = \frac{\tan 2 - \tan \theta}{1 + \tan 2 \tan \theta}$$

$$\approx \frac{\frac{2\epsilon}{a} - \frac{\epsilon}{u}}{1 + \frac{2\epsilon^2}{au}} \approx \frac{2\epsilon}{a} - \frac{\epsilon}{u} \ .$$

Since we already have $\tan \phi \approx \frac{\epsilon}{v}$ it follows that

$$\frac{\epsilon}{v} \approx \frac{2\epsilon}{a} - \frac{\epsilon}{u} \ ,$$

from which the result is immediate.

5. For the semi-circular mirror show that the cusp of the caustic $(\alpha = 0)$ corresponds to $D_\alpha^{\,2} g(\alpha,P) = 0$.

From (10) and (24) obtain

$$g(\alpha,P) = (a \cos \alpha - x)\tan 2\alpha - (a \sin \alpha - y) .$$

Differentiate twice to get

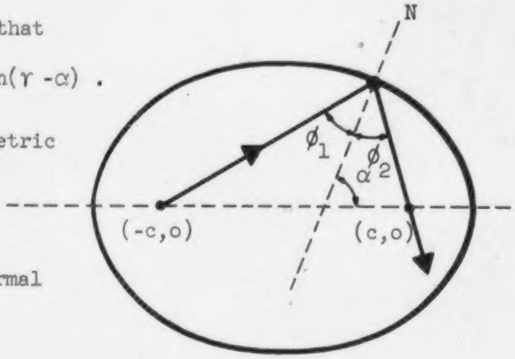
$$\begin{aligned} D_\alpha^{\,2}\, g(\alpha,P) = {} & 8(a \cos \alpha - x)\sec^2 2\alpha \tan 2\alpha \\ & - 2a \sec^2 2\alpha \sin \alpha - a \cos \alpha \tan 2\alpha \\ & + a \sin \alpha , \end{aligned}$$

which clearly vanishes when $\alpha = 0$.

6. Consider the elliptical reflector,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 , \qquad (a > b) .$$

Show that all the rays originating at one focus of the ellipse are reflected through the other focus. (The foci of the ellipse are the points $(\pm c,0)$ where $c = \sqrt{a^2 - b^2}$.)



Referring to the figure, show that

$$\tan \phi_1 = \tan(\alpha - \beta) = \tan \phi_2 = \tan(\gamma - \alpha) .$$

For this purpose use the parametric equations for the ellipse,

$$x = a \cos \theta , \; y = b \sin \theta$$

from which the slope of the normal $\vec{N}$ is

$$\tan \alpha = - \frac{x'}{y'} = \frac{a}{b} \tan \theta .$$

We have

$$\tan \beta = \frac{b \sin \theta}{a \cos \theta + c} , \quad \tan \gamma = \frac{b \sin \theta}{a \cos \theta - c} .$$

Enter these results in the expressions for $\tan \phi_1$ and $\tan \phi_2$ to obtain

$$\tan \phi_1 = c \sin \theta = \tan \phi_2 .$$

7. Verify analytically that the radius of curvature at a point P of a reflected eikonal for a semicircular mirror (Figure 15-2ℓ) is $R + \frac{a}{2}\cos\alpha$, where R is the distance from P along the ray to the mirror.

The radius of curvature is distance along the ray from P to the caustic, since the caustic is the evolute of the eikonal. Let I be the point of intersection of the ray with the mirror and J the place where it meets the caustic in Figure 15-2ℓ. We have $I = (-a\cos\alpha, a\sin\alpha)$, and, from the parametric equations for the caustic, (26) and (27),

$$J = \left(-\frac{a\cos\alpha}{2}[1 + 2\sin^2\alpha],\ a\sin^3\alpha\right);$$

whence

$$\begin{aligned}|\overline{IJ}| &= \left[\left\{a\cos\alpha\left(\tfrac{1}{2} - \sin^2\alpha\right)\right\}^2 + \left\{a\sin\alpha\cos^2\alpha\right\}^2\right]^{1/2}\\ &= \frac{a\cos\alpha}{2}.\end{aligned}$$

8. Find the edge ray caustic for a parabolic disk with edge given by

$$y^2 = 4px.$$

Use Equation (13) of Section 11-6 for the evolute. We have from Section 11-6, (12c), with y as the independent variable,

$$\hat{n} = \frac{(-2p, y)}{\sqrt{4p^2 + y^2}},$$

$$\kappa = \frac{4p_1^2}{(4p^2 + y^2)^{3/2}}.$$

Thus if (ξ, η) is the center of curvature corresponding to the point $(\frac{y^2}{4p}, y)$ on the parabola,

$$\begin{cases}\xi = \dfrac{y^2}{4p} - \dfrac{1}{2p}(4p^2 + y^2)\\[2ex] \eta = y - \dfrac{y}{4p^2}(4p^2 + y^2).\end{cases}$$

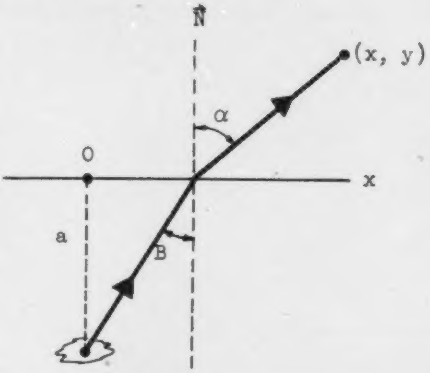
Eliminate y to obtain the cartesian form,

$$4(\xi + 2p)^3 = -p\eta^2,$$

a "semicubical parabola."

Solutions Exercises 15-3

1. Consider a point source under water $(\mu = \frac{4}{3})$ and the rays for which $\sin\beta < \frac{3}{4}$. Determine the virtual caustic for the rays refracted into air and show that one eikonal is an ellipse. (The apparently different positions of a small pebble in a dish of water as seen from different view points can be described in terms of this caustic). (Hint: Introduce the parameter $\cos\theta = \frac{\cos\alpha}{\cos\beta}$ where the angles α and β are the angles made with the surface normal in air and water, respectively.)



The problem is to determine the caustic of the rays on the air side of the interface. The equation of such a ray is

(i) $$y \tan\alpha - x + a\tan\beta = 0 .$$

We can express α in terms of β by means of Snell's Law

(ii) $$\sin\alpha = \mu \sin\beta$$

and replace β by θ using

(iii) $$\cos\theta = \frac{\cos\alpha}{\cos\beta} .$$

First, we observe in (i), from (ii) and (iii), that

(iv) $$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\mu\sin\beta}{\cos\theta\cos\beta} = \frac{\mu}{\cos\theta}\tan\beta .$$

Furthermore

$$\cos\theta = \frac{\cos\alpha}{\cos\beta} = \frac{\sqrt{1 - \mu^2\sin^2\beta}}{\cos\beta} ;$$

whence

$$1 - \mu^2\sin^2\beta = (1 - \mu^2) + \mu^2\cos^2\beta = \cos^2\beta\cos^2\theta .$$

and

$$\cos^2\beta = \frac{\mu^2 - 1}{\mu^2 - \cos^2\theta} .$$

From

$$\tan^2 \beta = \frac{1}{\cos^2 \beta} - 1 = \frac{\sin^2 \theta}{\mu^2 - 1}$$

we then obtain

$$\text{(v)} \qquad \tan \beta = \frac{\sin \theta}{\sqrt{\mu^2 - 1}} .$$

Entering (iv) and (v) in (i) we then have, in terms of the parameter θ ,

$$\text{(vi)} \qquad g(\theta,P) = \mu y \tan \theta - x\sqrt{\mu^2 - 1} + a \sin \theta = 0$$

where $P = (x,y)$. To find the caustic we determine the point (x,y) on each ray which satisfies both (vi) and

$$\text{(vii)} \qquad D_\theta\, g(\theta,P) = \frac{\mu y}{\cos^2 \theta} + a \cos \theta = 0 .$$

Thus the parametric equations for the caustic are

$$\text{(viii)} \qquad x = \frac{a}{\sqrt{\mu^2 - 1}} \sin^3 \theta \quad , \quad y = - \frac{a}{\mu} \cos^3 \theta ,$$

or, in cartesian form,

$$(x\sqrt{\mu^2 - 1})^{2/3} + (\mu y)^{2/3} = a^{2/3} .$$

Compare the solution to Exercises 11-6, Number 11, or Section 15-2, Equation (38) to obtain the elliptical eikonal

$$\text{(ix)} \qquad \frac{x^2}{a^2(\mu^2 - 1)} + \frac{y^2}{a^2\mu^2} = 1$$

or use Section 11-6, Equation 17(b) to obtain the involutes of the caustic. For this we observe that arclength σ along the caustic is given in terms of θ by

$$\left(\frac{d\sigma}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \frac{9a^2 \cos^2 \theta \sin^2 \theta[\mu^2 - \cos^2 \theta]}{\mu^2(\mu^2 - 1)} .$$

Take the square root and integrate to obtain

$$\sigma = \frac{a(\mu^2 - \cos^2 \theta)^{3/2}}{\mu\sqrt{\mu^2 - 1}} ,$$

where the constant of integration is omitted since it is provided for i the equation for the involutes. In terms of the parameter the involutes are given by

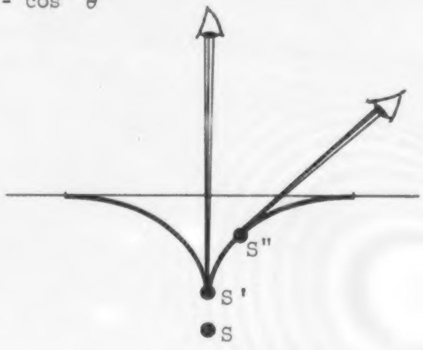
$$\xi = x + (c - \sigma)\frac{dx}{d\theta}\,\frac{d\theta}{d\sigma}$$

$$\eta = y + (c - \sigma)\frac{dy}{d\theta}\,\frac{d\theta}{d\sigma}\ ;$$

whence,

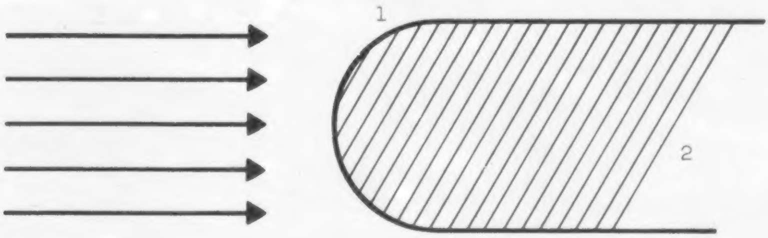
$$x = -a\sqrt{\mu^2 - 1}\,\sin\theta + \frac{c\mu\sin\theta}{\sqrt{\mu^2 - \cos^2\theta}}$$

$$y = -a\mu\cos\theta + \frac{c\sqrt{\mu^2 - 1}\,\cos\theta}{\sqrt{\mu^2 - \cos^2\theta}}\ ,$$



which gives the same ellipse as (ix) when $c = 0$. In viewing a source under the surface the narrow bundle of rays which reach the eye from the source emanate from the neighborhood of a virtual source on the caustic. When viewed from directly above, $\alpha = 0$, one sees the virtual source S' at a distance $\frac{a}{\mu}$ below the surface. For a larger angle of regard $0 < \alpha$ the virtual source S" is closer to the surface and somewhat displaced toward the observer, and in the limit as α approaches $\frac{\pi}{2}$, the source appears to be at the surface.

2. (a) Consider the two-dimensional problem of a set of parallel rays in medium 1 incident on a convex semicircle and strip of medium 2 as in the accompanying figure. Obtain the parametric equations for the caustic. Sketch the caustic for $\mu = \frac{4}{3}$.



Let a be the radius of the circle. As parameter take the angle α made by the incident ray with the normal. The equation of the refracted ray (see Figure (a)) is

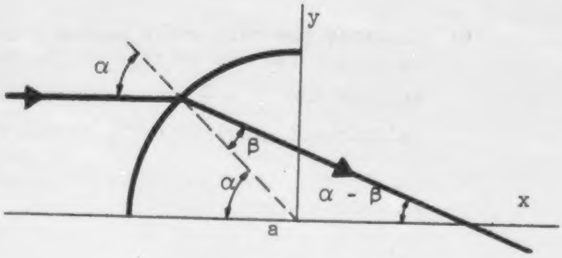


Figure (a)

(i) $$y = a \sin \alpha - (x + a \cos \alpha) \tan (\alpha - \beta)$$

where we may eliminate β by means of $\sin \beta = \frac{1}{\mu} \sin \alpha$. For simplicity, leave β in (i). Differentiate in (i) with respect to α and use $\frac{d\beta}{d\alpha} = \frac{\cos \alpha}{\mu \cos \beta}$ to obtain

$$0 = a \cos \alpha + a \sin \alpha \tan(\alpha - \beta) - \frac{(x + a \cos \alpha)}{\cos^2(\alpha - \beta)}\left(1 - \frac{\cos \alpha}{\mu \cos \beta}\right) .$$

Solve for x in this equation and substitute in (i) to obtain the parametric equations

$$\begin{cases} x = a\left[\dfrac{\mu \cos^2 \beta \cos(\alpha - \beta)}{\mu \cos \beta - \cos \alpha} - \cos \alpha\right] \\ y = -a\left[\dfrac{\mu \cos^2 \beta \sin(\alpha - \beta)}{\mu \cos \beta - \cos \alpha} - \sin \alpha\right] \end{cases}$$

The caustic may be plotted from these equations, or obtained by ray tracing as in Figure (b).

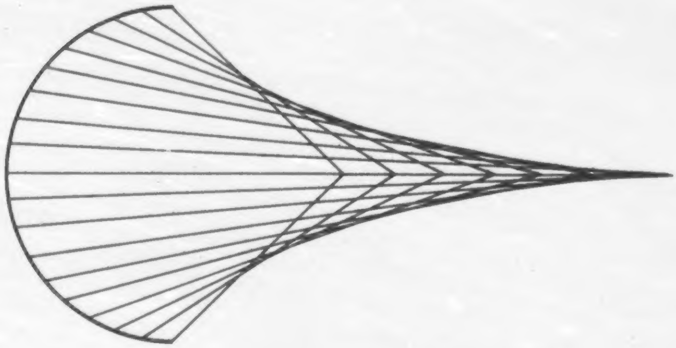


Figure (b)

(b) Consider the case where medium 2 is a circle. Obtain parametric equations for the caustic of the twice refracted rays. Sketch the caustic for $\mu = \frac{4}{3}$. (Is there a shadow? Try illuminating a cylindrical glass of water with a flashlight.)

For the second refracted rays we have the equation

(ii) $$y = a \sin\psi - (x - a\cos\psi)\tan(\alpha - \psi)$$

where $\psi = 2\beta - \alpha$; see Figure (c).

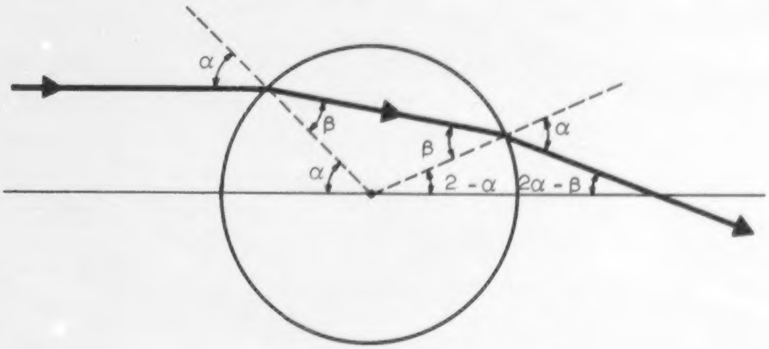


Figure (c)

Differentiate in (ii) with respect to ψ and use

$$\frac{d\psi}{d\alpha} = 2\,\frac{d\beta}{d\alpha} - 1 = \frac{2\cos\alpha}{\mu\cos\beta} - 1\ ;$$

to get

$$a\cos\psi - a\sin\psi\tan(\alpha - \psi) - \frac{2(x - a\cos\psi)(\mu\cos\beta - \cos\alpha)}{\cos^2(\alpha - \psi)(2\cos\alpha - \ \cos\beta)} = 0\ ;$$

whence, obtain the parametric equations

$$x = a\left[\frac{\cos\alpha\,(2\cos\alpha - \mu\cos\beta)\cos(\alpha - \psi)}{2(\mu\cos\beta - \cos\alpha)} + \cos\psi\right]$$

$$y = -a\left[\frac{\cos\alpha\,(2\cos\alpha - \mu\cos\beta)\sin(\alpha - \psi)}{2(\mu\cos\beta - \cos\alpha)} - \sin\psi\right]\ .$$

The second refracted rays are indicated in Figure (d).

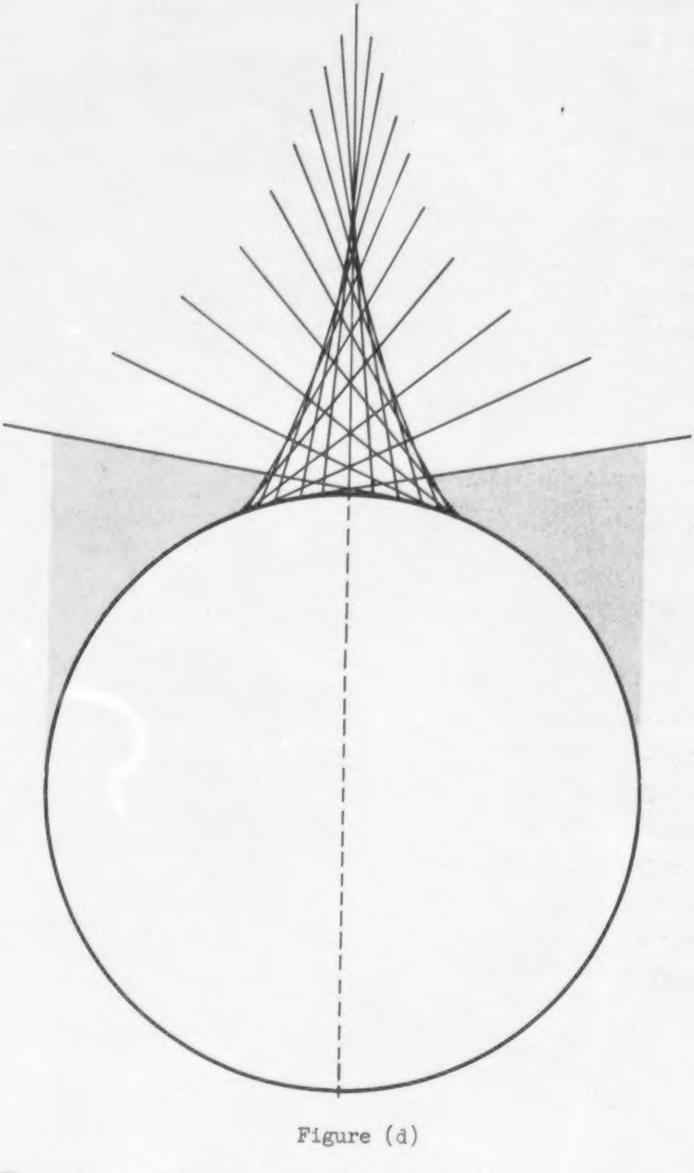
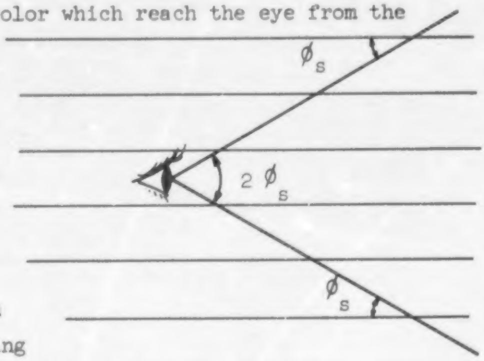


Figure (d)

There is a region of complete shadow (shaded in the figure) adjacent to a region of dim illumination outside the caustic.

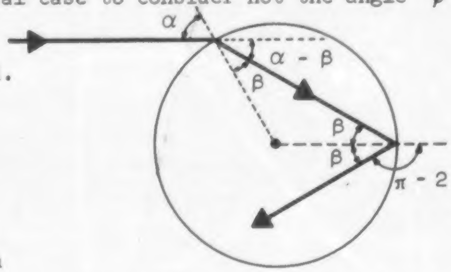
3. Consider sunlight illuminating a large number of drops over a very large volume of space and discuss how one will see the familiar arc of the rainbow.

The most intense rays of a given color which reach the eye from the rainbow make a fixed angle ϕ_s with the incident rays from the sun. Thus the back-scattered rays from a circular cone with vertex angle $2\phi_s$ and axis directed from the eye parallel to the incident solar radiation. When the sun is low on the horizon a person standing on a high building or a mountain can see the almost complete circle of the rainbow centered about the shadow of this head.



4. Show that ϕ_s is a maximum for the primary rainbow, and a minimum for the secondary bow. Using the fact that $\mu(\omega)$ increases as the colors go from red to blue, state the appearance of the primary and secondary arcs in space and the orders of the colors in the two cases. Derive (16).

It is more convenient for the general case to consider not the angle ϕ but the total angle ψ through which the incident ray is deflected. At the first refraction the ray is deflected through the angle $\alpha - \beta$. The ray next meets the surface at the same angle β with the normal and is deflected through the angle $\pi - 2\beta$ in a reflection or $\alpha - \beta$ in a refraction. If we consider a ray which is refracted outward after n internal reflections, the total deflection is



$$\psi = 2(\alpha - \beta) + n(\pi - 2\beta)$$

or

$$\psi = 2[\alpha - (n + 1)\beta] + n\pi \, . \tag{1}$$

Stationary ψ, corresponding to $\frac{d\psi}{d\alpha} = 0$, is then given by

$$\frac{d\beta}{d\alpha} = \frac{1}{n + 1} .$$

Since

(ii) $$\frac{d\beta}{d\alpha} = \frac{\cos\alpha}{\mu\cos\beta}$$

we obtain

(iiia) $$\mu\cos\beta = (n + 1)\cos\alpha .$$

With Snell's Law

(iiib) $$\mu\sin\beta = \sin\alpha$$

we may eliminate β by squaring in (iiia) and (iiib) and adding to obtain (16).

To determine the nature of the extremum ϕ_s for the primary and secondary bows, we observe from (i), (ii), (iiia) and (iiib) that $\left.\frac{d^2\psi}{d\alpha^2}\right|_{\alpha=\alpha_s} = 2\tan\alpha_s[1 - \frac{1}{(n + 1)^2}] > 0$ at the stationary point, or calculate in general

$$\frac{d^2\psi}{d\alpha^2} = -2(n + 1)\frac{d^2\beta}{d\alpha^2} = \frac{2(n + 1)(\mu^2 - 1)\sin\alpha}{\mu^3\cos^3\beta} > 0 ;$$

thus ψ_s is a minimum. For the primary rainbow, $\phi_s = \pi - \psi_s$, hence ϕ_s is a maximum; for the secondary bow, $\phi_s = \psi_s - \pi$, hence ϕ_s is a minimum.

Since $\phi_s \approx 42^\circ$ for the primary bow and $\phi_s \approx 51^\circ$ for the secondary bow, the secondary bow lies outside the primary bow. Furthermore ϕ_s is a decreasing function of μ for the primary bow, thus as ϕ decreases the colors range from the outside to the inside of the bow through red, yellow, blue in that order. For the secondary rainbow, ϕ_s is an increasing function of μ and the order of the colors is reversed.

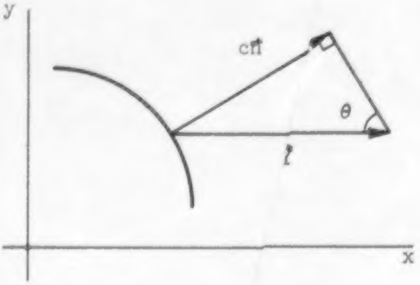
5. Sketch the variation of μ with height corresponding to the situations shown in Figure 15-3i and Figure 15-3j.

From (20), $\mu = c\sqrt{1 + (\frac{dx}{dy})^2}$. Take the x-axis vertically upward. In Figure 15-3i, let P be the highest point of the ray. As we proceed along the ray from any point toward P , x increases and $|\frac{dx}{dy}|$ decreases. Thus μ decreases with altitude.

If n is the unit normal vector to the curve then

$$\mu = \frac{c}{\sin\theta} = |\vec{\ell}| \quad \text{where} \quad |\vec{\ell}|$$

is constructed as in the adjacent figure. Thus it is easy to obtain a plot of $M : \mu \to x$ geometrically.



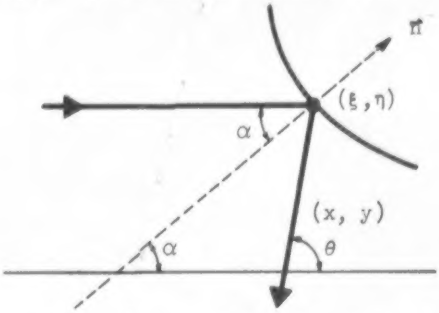
A similar argument for Figure 15-3j shows in that case that μ increases with altitude along the mirage forming ray.

Solutions Exercises 15-4

1. Derive the relation between flux and path length, Equation (16), from Equation (14).

Consider the paths to the point (x,y) by way of the convex mirror surface (see figure). Let $\vec{n} = (\cos\alpha, \sin\alpha)$ be the normal to the mirror. The tangent is $\vec{t} = (\sin\alpha, -\cos\alpha)$. If (ξ,η) is the point where the path meets the mirror, the path length is

$$L = \xi + R$$



as defined in Section 15-2, Equation (3). We have, as in Equation (4) of Section 15-2,

$$\text{(i)} \qquad L' = \xi'(1 + \cos\theta) + \eta' \sin\theta$$

where the prime may denote differentiation with respect to any parameter along the mirror curve. As we saw before, the condition that L be stationary, $L = L_H$, is that $L' = 0$, whence

$$\text{(ii)} \qquad \frac{\sin\theta}{1 + \cos\theta} = \tan\frac{\theta}{2} = -\frac{\xi'}{\eta'} = \tan\alpha \ .$$

Now let the parameter be arclength s along the mirror curve, so that the mirror curvature is, by definition, $\dot{\alpha} = \frac{d\alpha}{ds} = \frac{1}{a}$, where the dot indicates differentiation with respect to s . Now, $\dot{\xi} = \sin \alpha$, $\dot{\eta} = -\cos \alpha$ and, from

$$\xi = x + R \cos \theta \quad , \quad \eta = y + R \sin \theta$$

we obtain

(iiia)
$$\dot{\theta} = - \frac{1}{R}(\dot{\xi} \sin \theta - \dot{\eta} \cos \theta) \; ,$$

hence,

(iiib)
$$\dot{\theta} = - \frac{1}{R} \cos(\theta - \alpha) \; .$$

From (i), on differentiation with respect to s , we have

$$\ddot{L} = \ddot{\xi}(1 + \cos \theta) + \ddot{\eta} \sin \theta - (\dot{\xi} \sin \theta - \dot{\eta} \cos \theta)\dot{\theta} \; .$$

In this equation, enter the condition (ii) to obtain with the aid of (iiia) and (iiib)

$$\begin{aligned} \ddot{L}_H &= -(\ddot{\xi}\dot{\eta} - \dot{\xi}\ddot{\eta}) \frac{\sin 2\alpha}{\dot{\xi}} + R\dot{\theta}^2 \\ &= \frac{\dot{\alpha} \sin 2\alpha}{\sin \alpha} + \frac{\cos^2 \alpha}{R} \\ &= \frac{2}{a} \cos \alpha + \frac{\cos^2 \alpha}{R} \; ; \end{aligned}$$

whence

$$R\ddot{L}_H = \frac{2 \cos \alpha}{a}(R + \frac{a}{2} \cos \alpha) \; .$$

Enter this in (14) to get

$$F = \frac{\cos^2 \alpha}{R\ddot{L}_H} F_0 \; .$$

To get the other form of (16) let the prime denote differentiation with respect to α . Then

$$\dot{L} = L' \frac{d\alpha}{ds} = \frac{L'}{a}$$

whence,

$$\ddot{L} = (L'' \frac{d\alpha}{ds} + L' \frac{d^2\alpha}{ds^2}) \frac{d\alpha}{ds} \; .$$

Under the condition $L'_H = 0$ we have

$$\ddot{L} = \frac{L''}{a} ,$$

whence

$$F = \frac{a^2 \cos^2 \alpha}{RL''_H} F_0 .$$

Solutions Exercises 15-6

1. (a) Show that (19) is valid to first order in the parameter $\epsilon = \frac{d}{x}$.

In the notation of Figure 15-6e, we have

$$L_1 = \sqrt{(y - \tfrac{d}{2})^2 + x^2} \quad \text{and} \quad L_2 = \sqrt{(y + \tfrac{d}{2})^2 + x^2} ,$$

whence,

$$L_1 = x\sqrt{(\tan \theta - \tfrac{\epsilon}{2})^2 + 1} \quad \text{and} \quad L_2 = x\sqrt{(\tan \theta + \tfrac{\epsilon}{2})^2 + 1} .$$

Consequently, to first order in ϵ ,

$$L_1 \approx \frac{x}{\cos \theta} - \frac{\epsilon x \sin \theta}{2} , \quad L_2 \approx \frac{x}{\cos \theta} + \frac{\epsilon x \sin \theta}{2} ,$$

and

(i) $$L_2 - L_1 \approx \epsilon x \sin \theta = d \sin \theta ,$$

which yields (19) immediately.

(b) Show that the error in (20) is at most first order in ϵ .

The wave forms U_1 and U_2 in (20) are given by (16); namely, for $n = 1 , 2$

$$U_n = \frac{C}{\sqrt{kL_n}} e^{i(kL_n - \omega t)} .$$

Thus,

$$U = U_1 + U_2 = U_1 \left[1 + \sqrt{\frac{L_1}{L_2}} e^{ik(L_2 - L_1)} \right]$$

$$= U_1 \left[1 + \sqrt{\frac{L_1}{L_2}} e^{i\phi} \right].$$

To first order in ϵ we have

$$\sqrt{\frac{L_1}{L_2}} \approx 1 - \frac{1}{2}\,\epsilon \cos\theta \sin\theta \ ,$$

from which the result follows immediately.

2. Determine the extrema of the scattering amplitude $G(\theta)$ for a slit, Equation (33), by calculating the extrema of $f : x \to \frac{\sin x}{x}$. (The graph is given in Figure A 10, p. 665).

It is assumed that the function is extended continuously to $x = 0$ by $f(0) = 1$. From

$$f'(x) = \frac{x \cos x - \sin x}{x^2}$$

we see that the extrema occur when

(i) $$x = \tan x \ .$$

From

$$f''(x) = \frac{-x^2 \sin x - 2x \cos x + 2 \sin x}{x^3}$$

we see that the second derivative at an extremum satisfies

(ii) $$f''(x) = -\tan^2 x \,\frac{\sin x}{x} \ .$$

Furthermore, since $\tan x - x$ is increasing in each of the intervals $([n - \frac{1}{2}]\pi \ , [n + \frac{1}{2}]\pi)$ where $\tan x$ is defined, only one such extrema exists in each such interval. We need only consider $x \geq 0$ since $\frac{\sin x}{x}$ has even symmetry. For $x = 0$, $f(0) = 1$ is a maximum. For $x > 0$ the sign of $f''(x)$ at an extremum is determined from (ii) by the sign of $\sin x$. Since $x > 0$, an extremum can only occur when $\tan x > 0$, hence $x \in (n\pi , [n + \frac{1}{2}]\pi)$, $n = 1 , 2 , \ldots$. Thus, at an extremum, when n is even $f''(x) < 0$, and when n is odd $f''(x) > 0$. Consequently even n corresponds to a maximum and odd n to a minimum. To locate the extrema observe from (i) that for large x , $\tan x$ must also be large. Thus we may take as a first estimate for the n-th extremum

$$x \approx (n + \tfrac{1}{2})\pi \ .$$

To do better, set

$$\epsilon = (n + \tfrac{1}{2})\pi - x$$

and use the equation

$$\epsilon = \frac{\pi}{2} - \arctan[(n + \tfrac{1}{2})\pi - \epsilon]$$

to obtain an iteration scheme for ϵ .

Another procedure is to estimate ϵ by

$$\tan x = \frac{\cos \epsilon}{\sin \epsilon} = (n + \tfrac{1}{2})\pi - \epsilon \ .$$

Expand to second order in ϵ within

$$\cos \epsilon = [(n + \tfrac{1}{2})\pi - \epsilon]\sin \epsilon$$

to obtain

$$1 - \frac{\epsilon^2}{2} = [(n + \tfrac{1}{2})\pi - \epsilon]\epsilon \ ,$$

which yields

$$\epsilon = (n + \tfrac{1}{2})\pi \left\{1 - \sqrt{1 - \frac{2}{[(n + \frac{1}{2})\pi]^2}}\right\}$$

$$\approx \frac{1}{(n + \frac{1}{2})\pi} \ .$$

Thus, with a first correction,

(iii) $$x \approx (n + \tfrac{1}{2})\pi - \frac{1}{(n + \frac{1}{2})\pi} \ .$$

From (iii) we obtain successively for $n = 1 , 2 , 3 , \ldots$

$$x \approx 1.43\pi \ , 2.46\pi \ , 3.47\pi \ , \ldots$$

which correspond to the respective function values

$$f(x) = -0.22 \ , 0.13 \ , -0.09 \ , \ldots \ .$$

For large n , with $x \approx (n + \frac{1}{2})\pi$, we have

$$f(x) \approx (-1)^n \big/ (n + \tfrac{1}{2})\pi \ .$$

Solutions Exercises 15-7

1. Sketch vector diagrams for the first zero and first and second extrema of $\Gamma(\theta) = \frac{\sin(ka \sin\theta)}{ka \sin\theta}$ of Equation (4), where $\theta \geq 0$.

For simplicity take $a = 1$, and set $\gamma = k \sin\theta$. The vector diagram is traced out by the point (complex number P) given by

$$P = \int_{-1}^{\alpha} e^{i\gamma\eta} d\eta = \begin{cases} \frac{1}{i\gamma}[e^{i\gamma\alpha} - e^{-i\gamma}] & , \text{ for } \gamma \neq 0 \\ \alpha + 1 & , \text{ for } \gamma = 0 . \end{cases}$$

In terms of the parameter α , the point $P = (x,y)$ is given by

$$x = \frac{1}{\gamma}[\sin\gamma\alpha + \sin\gamma]$$

$$y = -\frac{1}{\gamma}[\cos\gamma\alpha - \cos\gamma]$$

for non-zero γ and $-1 \leq \alpha \leq 1$. For the first zero of $\Gamma(\theta)$, we have $\gamma = \pi$ and the point P traces out the circle $x^2 + (y + \frac{1}{\pi})^2 = \frac{1}{\pi^2}$ exactly once, Figure (a), so that the resultant Q is 0 . For the first maximum of $\Gamma(\theta)$ we have $\gamma = 0$ and the trace of P is the line segment, $x = \alpha + 1$, $y = 0$ for $-1 \leq \alpha \leq 1$, Figure (b). The second extremum (a minimum) is given by $\gamma \approx 1.43\pi$. The trace of P is an arc of the circle (the numbers are approximate)

$$(x + .216)^2 + (y + .052)^2 = (.222)^2$$

for a total central angle of $2\gamma \approx 2.86\pi$. (See Figure (c).) Thus the circle is completely lapped and the resultant Q is reached after the first lap.

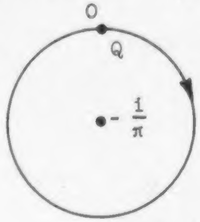


Figure (a)

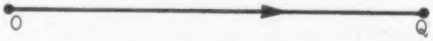


Figure (b)

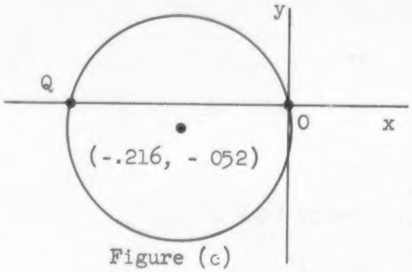


Figure (c)

Solutions Exercises 15-8

1. Verify that the plane waves

$$U = e^{\pm ikx\cos\alpha + iky\sin\alpha - i\omega t}$$

are solutions of (9) as claimed.

The result follows on addition of

$$\frac{\partial^2 U}{\partial x^2} = -k^2 \cos\alpha\, U ,$$

$$\frac{\partial^2 U}{\partial y^2} = -k^2 \sin\alpha\, U ,$$

$$-\frac{1}{v^2}\frac{\partial^2 U}{\partial t^2} = \frac{\omega^2}{v^2} U$$

where $\omega = kv$.

2. Show that any sufficiently differentiable function of the form

$$U = F(vt - x) + G(vt + x)$$

satisfies Equation (8).

Two derivatives are required. We have

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)U = \frac{\partial^2}{\partial x^2}F(vt-x) - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}F(vt-x) + \frac{\partial^2}{\partial x^2}G(vt+x) - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}G(vt+x) .$$

Note in the equation above that

$$\frac{\partial^2}{\partial x^2}F(vt - x) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}F(vt - x) = F''(vt - x)$$

and

$$\frac{\partial^2}{\partial x^2}G(vt + x) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}G(vt + x) = G''(vt + x) .$$

3. Verify that a solution of the wave equation (10) has the form (17), namely

$$E(x,y,z,t) = f(x,y,z)g(t) ,$$

only if

$$\frac{1}{f(x,y,z)}\Delta f(x,y,z) = \frac{1}{v^2 g(t)}\frac{d^2 g(t)}{dt^2} = \text{const.}$$

We have

$$0 = (\Delta - \frac{1}{v^2}\frac{\partial^2}{\partial t^2} f(x,y,z)g(t)$$

$$= \Delta[f(x,y,z)g(t)] - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}[f(x,y,z)g(t)]$$

$$= g(t)\Delta f(x,y,z) - \frac{f(x,y,z)}{v^2}\frac{\partial^2}{\partial t^2} g(t) ;$$

whence the result is immediate.

4. Obtain (43) and (44) as solutions of the linear system (38) - (41).

For brevity set

$$\alpha = e^{-ika} , \quad \beta = e^{iKa} .$$

Then (38) and (39) may be put in the respective forms

(iia) $$\alpha + \frac{R}{\alpha} = \frac{b_+}{\beta} + \beta b-$$

(iib) $$\frac{1}{2}(\alpha - \frac{R}{\alpha}) = \frac{b_+}{\beta} - \beta b-$$

where $R = g_-$ as in the text. Add and subtract (iia) and (iib) to get

(iiia) $$2b_+ = \frac{\beta}{Z}[\alpha(Z + 1) + \frac{R}{\alpha}(Z - 1)]$$

(iiib) $$2b_- = \frac{1}{\beta Z}[\alpha(Z - 1) + \frac{R}{\alpha}(Z + 1)] .$$

Put (40) and (41) in the respective forms

(iva) $$\alpha(\beta b_+ + \frac{b_-}{\beta}) = T$$

(ivb) $$\alpha Z(\beta b_+ - \frac{b_-}{\beta} = T .$$

Eliminate $\frac{T}{\alpha}$ in these equations to obtain

(v) $$\beta^4(Z - 1)b_+ - (Z + 1)b_- = 0 .$$

Enter the expressions for b_+ and b_- from (iiia) and (iiib) in (v) to obtain

$$R[\beta^4(Z - 1)^2 - (Z + 1)^2] + \alpha^2(Z^2 - 1)[\beta^4 - 1] ;$$

whence

$$R = \frac{-\alpha^2[1 - \beta^4]}{\frac{Z + 1}{Z - 1} - \frac{Z - 1}{Z + 1}\beta^4} = \frac{-Q\alpha^2[1 - \beta^4]}{1 - Q^2\beta^4} ,$$

which is Equation (43) in the present notation.

To obtain T , observe from (v) that

(vii) $$b_- = Q\beta^4 b_+ ,$$

from (iiia), that

$$b_+ = \frac{(Z + 1)\beta}{2Z\alpha}[\alpha^2 + QR]$$

or, from $2Z = (Z + 1)(1 + Q)$,

(viii) $$b_+ = \frac{\beta}{(1 + Q)\alpha}[\alpha^2 + QR] .$$

Enter (vii) and (viii) in (iva) to obtain

$$T = \alpha\beta b^+(1 + Q\beta^2)$$

$$= \frac{\beta^2(1 + Q\beta^2)(\alpha^2 + QR)}{1 + Q} .$$

Use the value of R given in (vi) in this expression for T to obtain

$$T = \frac{\alpha^2\beta^2(1 + Q\beta^2)(1 - Q^2)}{(1 + Q)(1 - Q^2\beta^4)} = \frac{1 - Q}{1 - Q\beta^2}\alpha^2\beta^2 .$$

5. Derive Equations (47) and (48).

The solution satisfies the Equation (25) in the two media, namely

(i) $$\left(\frac{d^2}{dx^2} + k^2\right)E_1 = 0 , \qquad (x < 0) ,$$

and

(ii) $$\left(\frac{d^2}{dx^2} + K^2\right)E_2 = 0 , \qquad (x > 0) ,$$

where $K = \mu k$. On the boundary, $x = 0$, we have the condition

(iii) $$E_1 = E_2 , \quad \frac{dE_1}{dx} = A\frac{dE_2}{dx} .$$

At infinity we impose the conditions appropriate to a planar scatterer:

(iva) $E_1 = E_i + E_s : E_i = e^{ikx} \quad , \quad E_s \sim ge^{-ikx}$ $(x \sim -\infty)$

(ivb) $E_2 \sim b\, e^{iKx} ,$ $(x \sim \infty)$

where the implication of (ivb) is that there is no wave from plus infinity. Thus, under condition (ivb) the solution of (ii) has the form

(v) $E_2 = be^{iKx} .$

We write, from (iva)

(vi) $E_s = ge^{-ikx} ,$ $(x < 0) ,$

and we need determine only the two constants b and g of (v) and (vi) to solve the problem. From the surface condition (iii) we have

(vii) $b = 1 + g$

and

(viii) $AKb = kg ,$

whence, in the notation of Number 4, with $Z = \frac{1 + Q}{1 - Q}$,

(ix) $b = \frac{Q - 1}{2Q} \quad , \quad g = - \frac{Q + 1}{2Q} .$

With these constants, we have in the two media

$$E_1 = E_i + E_s = e^{ikx} - \frac{1 + Q}{2Q} e^{-ikx} \quad , \quad E_2 = \frac{Q - 1}{2Q} e^{iKx} .$$